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ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ DEPARTMENT OF MATHEMATICS

1. ANDREOPOULOS PANAGIOTIS, BERSIMIS G. FRAGKISKOS, TRAGAKI ALEXANDRA
   MATHEMATICAL MODELLING OF MORTALITY DATA USING PROBABILITY DISTRIBUTIONS .... pp. 3-14
2. G. AVLOGIARIS
   LOCAL Φ-DIVERGENCES AND THEIR APPLICATIONS p.15
3. THANIPORN CHAYSRI
   ***M*-MATRICES AND THEIR EXTENSIONS** .... pp. 16-25
4. KONSTANTINA G. PALASKA
   VARIATIONAL SOLUTIONS OF BVPS VARIATIONAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR MONOTONE DISCRETE INCLUSIONS IN HILBERT SPACES .... p. 26
5. THEODOROS KASIOUMIS
   MINIMAL HYPERSURFACES IN R^4 .... p.27-30
6. LAZAROS MOYSIS AND NICHOLAS KARAMPETAKIS
   DIFFERENCE AND ALGEBRAIC EQUATIONS .... pp. 31-49
7. LAZAROS MOYSIS, IOANNIS KAFETZIS AND MARIOS POLITIS
   A DYNAMIC MODEL FOR HIV INFECTION .... pp. 50-58
8. CHRISTOS ONTI
   TOPOLOGICAL OBSTRUCTIONS OF ISOMETRIC IMMERSIONS ... p. 59
9. KLEANTHIS POLYMERAKIS
   TOPOLOGICAL OBSTRUCTIONS OF ISOMETRIC IMMERSIONS ... p. 60
10. CHRYSOSTOMOS PSAROUDAKIS
    RECOLLEMENTS OF DERIVED MODULE CATEGORIES .... p. 61
11. H. SEVASLIDOU
    THE ASSOCIATED LIE RING OF A FORMANEK-PROCESI GROUP .... pp. 62-64
12. CHRISTOS TATAKIS
    THE TRUE CIRCUIT CONJECTURE .... p. 65
13. ILIAS TERGIAKIDIS
    THE RICCI FLOW IN DIMENSION THREE .... pp. 66-90
14. FOTIS THOMOS
    CHAOTIC LINEAR OPERATORS .... p. 91
15. CHARALAMPOS ZINOVIADIS
    HIERARCHY AND EXPANSIVENESS IN 2-DIMENSIONAL SUBSHIFTS OF FINITE TYPE .... pp. 92-94

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MATHEMATICAL MODELLING OF MORTALITY DATA USING PROBABILITY DISTRIBUTIONS

ANDREOPOULOS PANAGIOTIS, BERSIMIS. G. FRAGKISKOS, TRAGAKI ALEXANDRA

Abstract. A number of different distributions describing age-related mortality have been proposed. The most common ones, Gompertz and Gompertz - Makeham distributions have received wide acceptance and describe fairly well mortality data over a period of 60-70 years, but generally do not give the desired results for old and/or young ages. This paper proposes a new mathematical model, combining the above distributions with Beta distribution. Beta distribution was chosen for its flexibility on age-specific mortality characteristics. The proposed model is evaluated for its goodness of fit and showed sufficient predictive ability for different population sub-groups. The scope of this work is to create sufficient mortality models that could also be applied in populations other than the Greek, based on appropriate parameter detection (e.g. Maximum Likelihood) and to come up with a methodology comparison. Our analysis relied on 2011 disease data tabulated by age (5-year groups) and sex provided by the Greek Statistical Authority (ELSTAT) as part of the natural movement of the population of Greece. Deaths are decomposed by different causes, endogenous or external causes of death. According to our preliminary findings, the proposed mortality model (ANBE) shows satisfactory results on appropriate evaluation criteria (AIC, BIC). This paper presents some of the statistical properties of ANBE model.

1. Introduction

Demography is a data driven field of scientific research: data help identifying spatial and temporal trends and fluctuations and is the key-element on which optimal policy decisions are based. A lot of work has been carried out to construct and optimize, among others, probabilistic mortality models that describe a population's mortality. The aforementioned models are constructed by using various physical or demographic quantities such as births, deaths, aging and marital status (single, married, widow etc), as independent variables. In this work calculated mortality rates, which are available in tabular form called mortality tables, were used. A mortality table is a series of (annually, monthly, weekly, etc) death probabilities $q_x, q_{x+1}, ..., q_{\omega+1}$ from a minimum age until a maximum theoretical biological age threshold, where ω symbolize the marginal age i.e. the age beyond which no individual can't survive theoretically and q_x symbolize the probability, of an individual

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aged x, to die in time interval [x, x+1) for x=a, a+1,, ω-1. The probability of an aged x individual to die in this interval is given by relation 1.1:

$$q_x = \frac{d_x}{l_x} \tag{1.1}$$

where d_x symbolizes the number of deaths in the interval [x,x+1), and is given by the difference of relation 1.2:

$$d_x = l_x - l_{x+1} \tag{1.2}$$

where l_x symbolizes the multitude of individuals that are alive in the age of x. Mortality tables are used in everyday life for analyzing populations' mortality and estimating the corresponding mortality rate. In Mortality tables certain functions (Johnson and Johnson, 1980) are contained such as $l_x, d_x, q_x, p_x, x = 1, 2, , \omega$, where p_x represents the conditional probability of survival meaning the probability of an individual of age x not to die in the interval [x,x+1). By expanding the previous definition, he probability an individual of age x to survive n additional years, is given by relation 1.3:

$$P_x = \frac{l_{x+n}}{l_x} \tag{1.3}$$

In addition, function e_x symbolizes the number of years that an individual of age x is expected to live and is given by relation 1.4:

$$e_x = \frac{T_x}{l_x} \tag{1.4}$$

2. Focus on target

In literature, there are (Benjamin, P. and Polland, J.H., 1980) mathematical models of mortality that use special distributions (Gompertz, Makeham), suitable for describing demographical data. But all so far fail to include in their analysis at the same time specific fields such as age, accidents and diseases. But, the combination of the previous distributions with the Beta distribution, which takes appropriate customized form, constructs mathematical models with better fit and also sufficient predictive ability for different data sets (data deaths or diseases).

There are two basic methods for constructing a mortality model that attempts to represent and assess the real mortality pattern:

a) In the first method, a mortality table is constructed and a mortality time series is produced from its elements.

b) In the second method, the mortality pattern is described by a function based on some known probabilistic distributions.

In the case of parametric methods, the adaptation of a mathematical function in the initial estimations of mortality rates or the mortality intensity is desirable, in order to express the relation that exists between them. The normalization is conducted either by adjusting a function in all ages or more functions to data incrementally. Non-parametric methods are applied to mortality tables by combining data in different values of age x. The main objective of these methods is to smooth the values of mortality tables by using mortality models, generated by methods such as generalized linear models or splines methods.

3. Mortality model selection

One of the main objectives of Demography is the description of the mechanisms that describe mortality. The mortality models, as it was mentioned before, contain various demographic measures such as the mortality intensity, symbolized as μ_x and the mortality probability, symbolized as q_x. According to Hatzopoulos [10], a satisfying model should have the right theoretical background in order to give better interpretations. In addition, a model should give the best results with the minimum number of parameters. As an evolution of the above laws, a mortality law was formulated by Gompertz, about one hundred years later, by using the corresponding probability distribution. The probability density function of Gompertz distribution with parameters a and b is given in relation 3.1:

$$f(x) = ae^{bx}e^{-\frac{a}{b}(e^{bx}-1)}, x \geq 0, a, b > 0 \tag{3.1}$$

According to this law, mortality intensity μ_x grows exponentially according to the type of relation: 3.2:

$$\mu_x = ae^{bx} \tag{3.2}$$

where a and e^b are positive parameters and e^b takes values close to 1.09 (Lytrokapi [13]). The logarithm of the mortality intensity is the same as the log link generalized linear models used (Hatzopoulos, 1997). This law often describes satisfactorily enough the empirical data of population mortality, at least in the interval of 60-70 years, but generally does not give the desired results in young ages. The basic disadvantage of Gompertz law is that only takes into consideration the systematic "physiologic deterioration" and ignores the effect of accidental element. The initial mortality models always aim to adapt data better to reality, leading to the proposition of the "1st law of Makeham, which is give in relation 3.3:

$$\mu_x = ae^{bx} + \lambda \tag{3.3}$$

where λ is a constant parameter that expresses the randomness of deaths due to deteriorating health or accidents. The introduction of the third parameter λ renders the Makeham law more flexible in the depiction of empirical data. Then, another law was found, known as the "2nd law of Makeham" or "Generalized Makeham law, which is given by relation: 3.4:

$$\mu_x = ae^{bx} + hx + \lambda \tag{3.4}$$

where h is an unknown parameter. A lot of mortality models exist in the Demography literature and investigators select the appropriate model that adopts better real data. In addition, a combination of already known distributions (Exponential, Gamma, Beta distribution, etc.) with similar mortality laws is possible to be carried out.

4. Building the mortality model

The mortality rates were built, by using data emanating from the Greek Statistical Service related to the Greek mortality data from year 2011. Deaths are decomposed by different causes, endogenous or external causes of death. The real mortality pattern is described satisfactorily by the models that were mentioned before. Then, the models are adjusted to the original data, in order to estimate the unknown parameters by using the maximum likelihood method and the least squares method.

4.1. **Maximum likelihood.** Let the number of deaths at age x, to follow the binomial distribution with parameters l_x and q_x, where l_x is the number of individuals in danger in the age of x and q_x is the respective actual mortality rate, assuming that deaths in various age groups are independent to each other. The likelihood is given in relation 4.1:

$$L = \prod_{x=1}^{n} \binom{l_x}{d_x} {q_x}^{d_x} (1 - q_x)^{l_x - d_x} \tag{4.1}$$

The unknown element in the above likelihood is q_x. In order to estimate the maximum likelihood for unknown parameters of the model, the natural logarithm of L is obtained and the corresponding mathematical form is given in equation: 4.2:

$$logL = \sum_{x=1}^{n} [log \binom{l_x}{d_x} + d_x log q_x + (l_x - d_x) log(1 - q_x)] \tag{4.2}$$

4.2. **Least Squares.** Suppose, the initial mortality rate estimation at age x is the value that is adjusted in this age. The choice of the suitable model $m(x)$ is conducted by adjusting estimated values as close as possible to the observed values. In order to achieve the minimization of the distance between real and estimated values, the method of weighted least squares is applied, which is given by relation 4.3

$$\sum_{x=1}^{n} w_x [q_x - m(x)]^2 \tag{4.3}$$

5. Beta distribution and mortality Models

The Gompertz-Makeham distribution expresses the law stating that mortality rate represents the sum of an independent component λ and a component that depends on the age which increases exponentially with time. The Gompertz-Makeham law satisfactorily describes the dynamic human age, but there is a loss of accuracy between 30 and 80 years. In literature, studies report that death rates after the age of 80 years increases more slowly, a phenomenon called as decelerating mortality (Gavrilov, L., Gavrilova, N. [6]). The probability density function of Gompertz-Makeham distribution (pdf) is given as follows in relation 5.1

$$f(x) = (ae^{bx} + \lambda) e^{-\lambda x - \frac{a}{b}(e^{bx} - 1)}, a > 0, b > 0, \lambda > 0, x \geq 0 \tag{5.1}$$

The corresponding hazard function of Gompertz-Makeham is given in relation 5.2:

$$h(x) = ae^{bx} + \lambda, \lambda > 0, x \geq 0 \tag{5.2}$$

The corresponding cumulative distribution function of Gompertz-Makeham is given in relation 5.3:

$$F(x) = 1 - e^{-\lambda x - \frac{a}{b}(e^{bx} - 1)}, a > 0, b > 0, \lambda > 0, x \geq 0 \tag{5.3}$$

The Gompertz-Generalized Makeham distribution has the following probability density function 5.4:

$$f(x) = (2\kappa\theta\xi x + \lambda + \theta e^{\xi x})[e^{-(\kappa\theta\xi x^2 + \lambda x + \frac{\theta}{\xi}(e^{\xi x} - 1)}], \kappa, \theta, \xi, \lambda > 0, x \geq 0 \tag{5.4}$$

and the corresponding cumulative distribution function is given as follows in relation 5.5:

$$F(x) = 1 - e^{-(\kappa\theta\xi x^2 + \lambda x + \frac{\theta}{\xi}(e^{\xi x} - 1)}], \kappa, \theta, \xi, \lambda > 0, x \geq 0 \tag{5.5}$$

The Beta distribution has been used widely in the past years in a variety of scientific fields. Beta distribution is capable of being adapted in a variety of data by each researcher, always under the appropriate transformation. In regression models, Beta distribution is used by researchers for modeling data taking values in the interval (0, 1). For combining Beta distribution with the aforementioned mortality models, Generated Beta distribution is used (M. Zografos, N. Balakrishnan [17]). The corresponding probability density function is given by 5.6:

$$g(x) = \frac{f(x)}{B(\alpha,\beta)}[F(x)]^{\alpha-1}[1-F(x)]^{\beta-1}, \alpha,\beta > 0, x \geq 0 \tag{5.6}$$

where $F(x) = 1-S(x)$ and $h(x) = f(x)/S(x)$ the corresponding hazard function. From the family of Beta distribution applies relation 5.7:

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{(\beta-1)}dt \tag{5.7}$$

Generally, Gompertz distribution is a flexible distribution that is asymmetrical right or left. In addition, this distribution is a generalization of the exponential distribution and is widely used in many applicant problems, especially in life data analysis (Johnson, Kotz and Balakrishnan 1995). The combination of Generated Beta distribution with the Gompertz distribution leads to Beta-Gompertz distribution (Ali Akbar Jafari, SaeidTahmasebi, MoradAlizadeh, (2014)) with probability density function 5.8:

$$f(x) = \frac{ae^{bx}e^{-\frac{a}{b}(e^{bx-1})}}{B(\alpha,\beta)}(1-e^{-\frac{a}{b}(e^{bx-1})})^{\alpha-1}(e^{-\frac{a}{b}(e^{bx-1})})^{b-1}, \alpha > 0, b > 0 \tag{5.8}$$

Combination of Generated Beta distribution with the Gompertz-Makeham distribution leads to Beta-Gompertz-Makeham distribution (Chukwu A. U., Ogunde A. A. [4]) with probability density function 5.9:

$$f(x) = \frac{(ae^{bx}+\lambda)e^{-\lambda x-\frac{a}{b}(e^{bx-1})}}{B(\alpha,\beta)}(1-e^{-\lambda x-\frac{a}{b}(e^{bx-1})})^{\alpha-1}(e^{-\frac{a}{b}(e^{bx-1})})^{b-1} \tag{5.9}$$

6. The proposed ANBE model

As pre mentioned, this work presents a mixed version of the generalized Gompertz-Makeham distribution with Beta distribution that is quite flexible in respect to its parameters. The name of this is ANBE model. Beta Gompertz Generalized Makeham (BGGM or ANBE) distribution is a decreasing or increasing or bathtub-shaped density function according to its six parameters. The proposed distribution (BGGM or ANBE), applied in real mortality data and especially in diseases, succeeds higher values in the corresponding fit criteria (Log-likelihood, AIC, BIC) and seems to fit better in real data compared to other distributions, such as Gompertz, Gompertz Makeham, Beta Gompertz, Beta Gompertz Makeham. Available data were in the form (d_x, l_x), where d_x represents the number of deaths (or diseases) at age x=1,2,,n and l_x represents the total of individuals up to that age x. The abundant mortality index in that event is given from equation of relation 6.1:

$$q_x = \frac{d_x}{l_x} \tag{6.1}$$

6.1. **General Linear Method (GLM) and Beta distribution.** Mortality rates are smoothed, through GLM, based on the assumption that the response variable D_x follows a binomial distribution (Haberman [8]). In addition, mortality intensity μ_x is assumed to be a constant in ages intervals (x,x+1], therefore is symbolized with $\mu_{x+1/2}$ (Haberman and Pitacco [9], Renshaw et al. [16]). It is noted that ages should be separated in subintervals and in each one of these a different model is adapted. This method is well known as smoothing through splines methods. In this work, a spline function is the age function x=1,2,...,n that gives the smoothed values. The proposed BGGM mortality model (Beta Gompertz Generalized Makeham) based in mixing Beta, Gompertz and generalized Makeham distributions has probability density function, as follows 6.2:

$$f(x) = \frac{(2\kappa\theta\xi x + \lambda + \theta e^{\xi x})[e^{-(\kappa\theta\xi x^2+\lambda x+\frac{\theta}{\xi}(e^{\xi x}-1)}]}{B(\alpha,\beta)} \times \\ \times (1 - e^{-(\kappa\theta\xi x^2+\lambda x+\frac{\theta}{\xi}(e^{\xi x}-1)})^{\alpha-1}(e^{-(\kappa\theta\xi x^2+\lambda x+\frac{\theta}{\xi}(e^{\xi x}-1)})^{b-1}, \\ \kappa,\theta,\xi,\lambda,\alpha,b > 0 \quad (6.2)$$

6.2. **Some Statistical Properties of ANBE model.** In this paragraph function $f(x)$ is verified to be a probability density function, as well as the cumulative density $G(x)$ and hazard rate $H(x)$ functions of the proposed BGGM distribution are investigated. In addition, the asymptotic behavior of probability density function BGGM is examined for specific values of its parameters.

Corollary 6.1. *Let $f(x)$ be the pdf of a variable x that follows the BGGM distribution. The asymptotic behavior of function for different values of its parameters is given below:*

i. If $\alpha = 1$ then $\lim_{x\to 0^+} f(x) = b(\lambda + \theta)$, i.e., when variable x tents to zero and parameter α is equal to one, the corresponding pdf of BGGM takes a constant value, depending to its parameters.

ii. If $\alpha > 1$ then $\lim_{x\to 0^+} f(x) = 0$, i.e., when variable x tents to zero and parameter α is greater than one, the corresponding pdf of BGGM takes a zero value.

iii. If $0 < \alpha < 1$ then $\lim_{x\to 0^+} f(x) = +\infty$, when variable x tents to zero, the corresponding pdf of BGGM is non decreasing and tends to infinity.

iv. $\lim_{x\to +\infty} f(x) = 0$, when variable x tents to infinity, the corresponding pdf of BGGM takes a zero value.

7. Application on different data set

In this section, obtaining actual mortality data from the Greek Statistical Service for the year 2011 for Greece, the direct comparison of proposed distribution (BGGM) with a lot of known distributions such as Gompertz, Gompertz - Makeham, Beta - Gompertz and Beta-Gompertz - Makeham distributions is attempted. As mentioned, the analysis relied on 2011 disease data tabulated by age (5-year groups) and sex provided by the Greek Statistical Authority (ELSTAT) as part of the natural movement of the population of Greece. Deaths are decomposed by different causes, endogenous or external causes of death. The data modelling and the corresponding distributions functions, were conducted by using open code R (www.r-project.org) and corresponding packages of algorithms by CRAN digital library (Comprehensive R Archive) http://cran.r-project.org. Mortality rates for

	Male			Female		
Mortality Models	Log-likelihood	AIC	BIC	Log-likelihood	AIC	BIC
Gompertz	-387.0	778.1	779.9	-155.7	315.4	317.1
Gompertz Makeham	-354.1	714.3	717.0	-142.3	290.7	293.4
Beta Gompertz	-116.6	241.2	244.7	-109.7	227.4	231.0
Beta Gompertz Makeham	-114.4	238.9	243.4	-107.9	225.9	230.4
Beta Gompertz Generalized Makeham (ANBE)	-99.7	217.5	225.5	-92.1	202.2	210.2

Table 1. Statistical criteria for controlling Suitability distributions - Cancer

	Male			Female		
Mortality Models	Log-likelihood	AIC	BIC	Log-likelihood	AIC	BIC
Gompertz	-64.0	132.1	135.8	-83.1	170.3	172.1
Gompertz Makeham	-62.1	132.3	136.3	-58.5	123.1	131.7
Beta Gompertz	-60.9	134.0	133.9	-57.9	123.0	129.6
Beta Gompertz Makeham	-58.0	131.9	132.5	-57.5	125.1	125.8
Beta Gompertz Generalized Makeham (ANBE)	-52.4	110.9	113.5	-52.8	123.7	122.8

Table 2. Statistical criteria for controlling Suitability distributions - Endocrine

exogenous and endogenous factors were used, for men and women. The number of deaths at age x, d_x, is based on a sample of size l, where l is the total amount of deaths. Consequently, the initial assessment of the mortality rates is considered as follows 7.1:

$$u_x = \frac{d_x}{l_x} \tag{7.1}$$

In the following figures (1-6), mortality rate models are presented according to the distributions used, as well as, sex and region. Red line corresponds to the proposed ANBE (BGGM) model that seems to fit better in the actual data, both for men and women, as well as for different endogenous causes of death. At the figures, the number 5 corresponds to 25 years old, the number 10 corresponds to 50 years old, the number 15 corresponds to 75 years old and the number 20 corresponds to 100 years old at the x axis.

Results in Tables 1, 2 and 3 indicate that the ANBE model appears to have the best fit for the three major causes of death (cancer, endocrine, nutritional and metabolic diseases and circulatory system causes) examing both men and women. This is verified based on all three criteria: the AIC, BIC and log-likelihood criterion.

	Male			Female		
Mortality Models	Log-likelihood	AIC	BIC	Log-likelihood	AIC	BIC
Gompertz	-1020.9	2045.8	2047.6	-1008.3	2020.7	2022.5
Gompertz Makeham	-485.7	977.5	980.2	-233.0	472.0	474.7
Beta Gompertz	-120.6	249.3	252.9	-106.3	220.6	224.1
Beta Gompertz Makeham	-119.4	248.8	250.3	-103.4	216.9	221.3
Beta Gompertz Generalized Makeham (ANBE)	-104.3	226.6	234.7	-81.0	180.1	188.1

Table 3. Statistical criteria for controlling Suitability distributions - Circulatory system

	Male			Female		
Mortality Models	Log-likelihood	AIC	BIC	Log-likelihood	AIC	BIC
Gompertz	-268.5	541.0	542.7	-102.7	209.5	211.3
Gompertz Makeham	-251.2	508.4	511.1	-86.2	178.4	181.1
Beta Gompertz	-102.4	212.9	216.5	-73.9	155.8	159.4
Beta Gompertz Makeham	-102.1	210.9	219.4	-73.8	157.6	162.1
Beta Gompertz Generalized Makeham (ANBE)	-90.9	199.8	207.8	-67.1	152.2	160.2

Table 4. Statistical criteria for controlling Suitability distributions - All external causes

In the following figures (7-12), mortality rate models are presented according to the distributions used, as well as, sex and region. Red line corresponds to the proposed ANBE (BGGM) model that seems to fit better in the actual data, both for men and women, as well as for different external causes of death

8. Conclusions

The estimators of mortality as described earlier, i.e. indicative mortality rates (crude mortality rates) are subjected to the sampling errors, allowing a non-smooth progress from age to age. Suppose that these errors are only due to random variability that is inherent in the finite sample observe. Specifically, increasing the sample size leads to a decrease of errors and the target prices have smooth progress along ages. Thus, mortality can be assumed to be a continuous and smooth function of age. The process of eliminating random errors is known as normalization (graduation). Smoothing, practical means to remedy the lack of possibility of existence of an infinite sample size with an alternative assessment of mortality rates as accurately as possible. Real mortality data is approached satisfactorily by the

	Male			Female		
Mortality Models	Log-likelihood	AIC	BIC	Log-likelihood	AIC	BIC
Gompertz	-188.0	380.1	381.9	-61.9	127.8	129.6
Gompertz Makeham	-179.9	365.9	368.6	-60.0	126.1	128.8
Beta Gompertz	-87.5	185.0	189.5	-57.9	123.9	127.5
Beta Gompertz Makeham	-87.5	183.1	186.7	-57.4	124.9	126.3
Beta Gompertz Generalized Makeham (ANBE)	-75.9	169.9	177.9	-46.8	111.6	119.6

Table 5. Statistical criteria for controlling Suitability distributions - Car accidents

	Male			Female		
Mortality Models	Log-likelihood	AIC	BIC	Log-likelihood	AIC	BIC
Gompertz	-80.7	165.4	167.2	-82.2	168.5	170.3
Gompertz Makeham	-67.4	140.8	148.8	-64.7	135.4	146.4
Beta Gompertz	-66.1	140.2	147.8	-62.5	133.1	139.6
Beta Gompertz Makeham	-66.0	140.1	146.6	-62.5	131.1	138.1
Beta Gompertz Generalized Makeham (ANBE)	-60.9	139.8	143.7	-60.2	130.4	136.7

Table 6. Statistical criteria for controlling Suitability distributions - Mental and behavioral disorder

corresponding distributions that were mentioned before (Figures 1-12). The proposed ANBE distribution (red color) approximates data much better for both men and women, as well as, for different endogenous and external causes of death. By using goodness of fit tests, the comparison of proposed distribution with other distributions is attempted. The corresponding results in the tables above (1-6) and the corresponding criteria values, such as the log-likelihood (Loglik), AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) show that ANBE model has the best fit to the different data set. As reported on Tables and Figures, different model selection criteria have led to the same conclusions, showing that this model has the best fit for all different causes of death and for both sexes. This is a very promising sign that our new ANBE distribution achieves better values compared to other distributions when applied on Greek mortality data. It remains to be examined if it can be applied on different datasets and various time periods.

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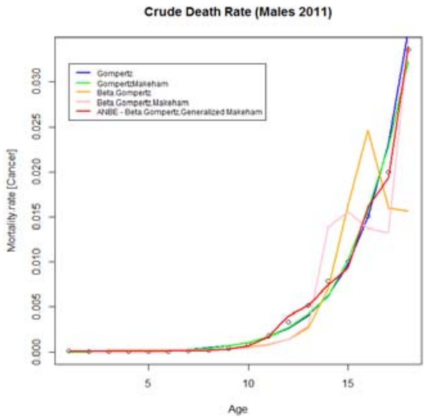

FIGURE 1. ANBE [BGGM] (All Cancers) - Males

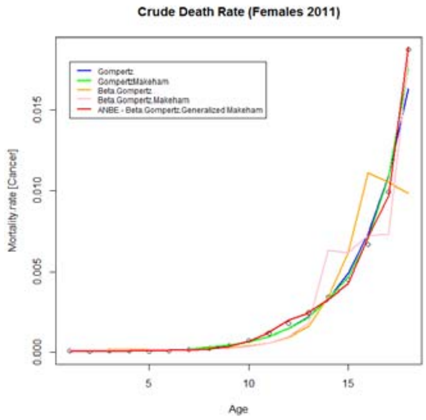

FIGURE 4. ANBE [BGGM] (All Cancers) - Females

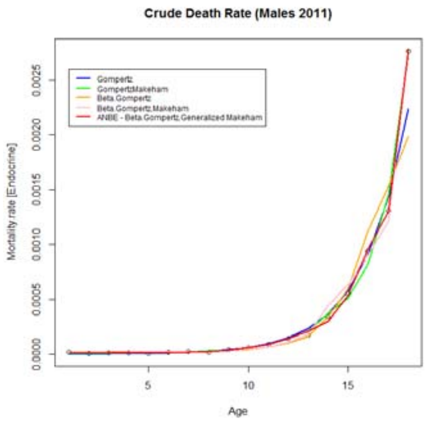

FIGURE 2. ANBE [BGGM] (Endocrine, nutritional and metabolic diseases) - Males

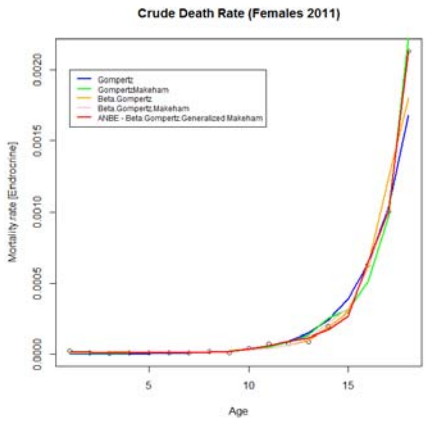

FIGURE 5. ANBE [BGGM] (Endocrine, nutritional and metabolic diseases) - Females

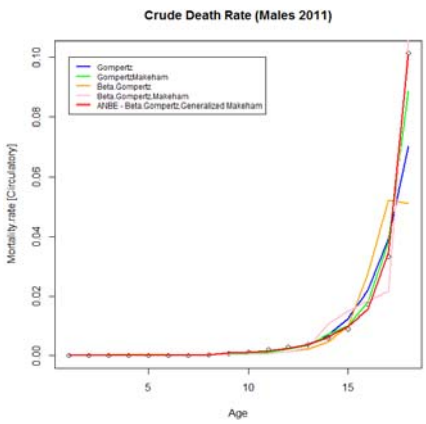

FIGURE 3. ANBE [BGGM] (Diseases of circulatory system) - Males

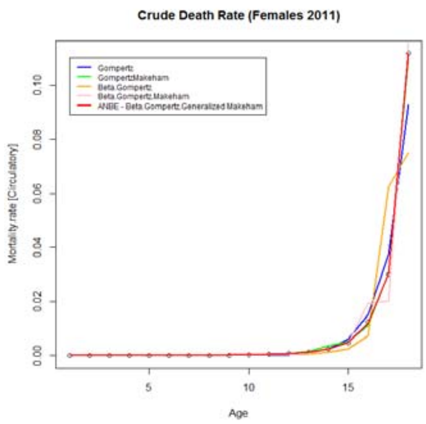

FIGURE 6. ANBE [BGGM] (Diseases of circulatory system) - Females

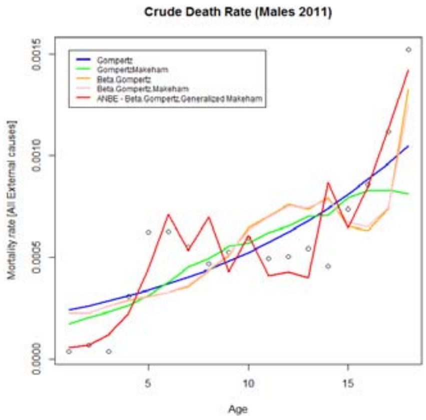

FIGURE 7. ANBE [BGGM] (All External causes)-Males

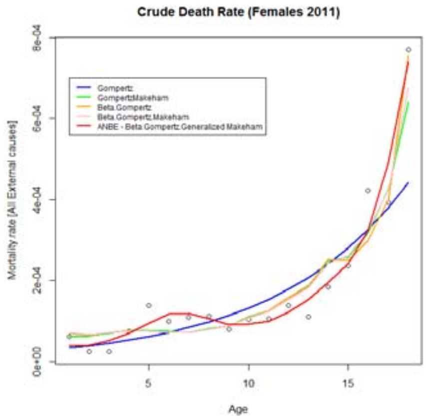

FIGURE 10. ANBE [BGGM] (All External causes) - Females

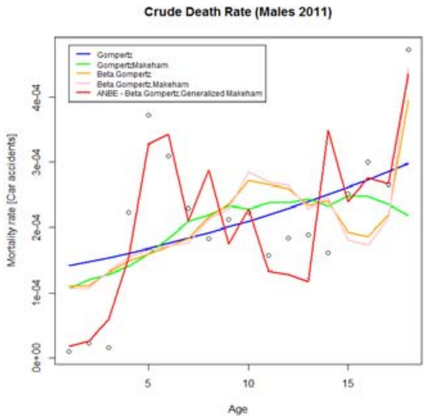

FIGURE 8. ANBE [BGGM] (Car accidents) - Males

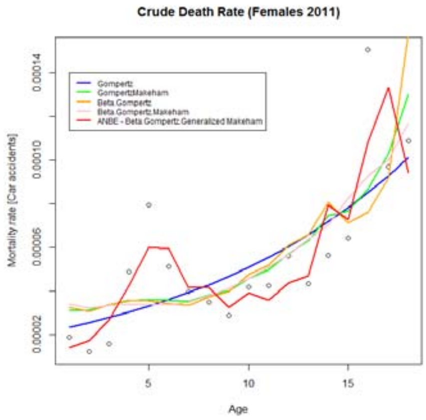

FIGURE 11. ANBE [BGGM] (Car accidents) - Females

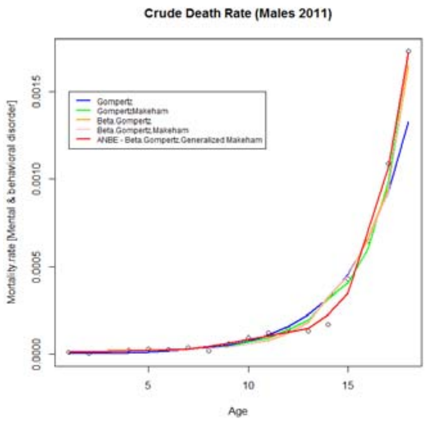

FIGURE 9. ANBE [BGGM] (Mental and behavioral disorder) - Males

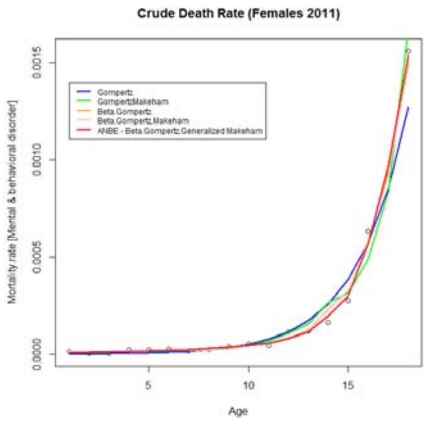

FIGURE 12. ANBE [BGGM] (Mental and behavioral disorder) - Females

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LOCAL Φ-DIVERGENCES AND THEIR APPLICATIONS

G. AVLOGIARIS[1]

ABSTRACT. The concept of ϕ-divergence between two probability measures or between the respective probability distributions, introduces a broad class of statistical pseudo-distances. This talk will focus on the definition of divergence measures in a local setting, that is, on the definition of divergence measures between two probability distributions in a subset of their joint domain. In this context, the definition and properties of local ϕ-divergence will be presented and analytical expressions of the local ϕ-divergence between two members of the exponential family of distributions will be given. Applications of the proposed measures of local divergence in testing statistical hypotheses will be also discussed.

This is joint work with A. Micheas[2] and K. Zografos[1]

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$M-$MATRICES AND THEIR EXTENSIONS

THANIPORN CHAYSRI

ABSTRACT. The foundations of what today is called M-matrix was originally chosen by Alexander Ostrowski in reference to Hermann Minkowski, which consider the square matrices of the form $A = sI - B$. In this paper, we first study the matrices of the form $A = sI - B$, where B is entrywise nonnegative and $0 \leq \rho(B) \leq s$, which called $M-$matrices. Then we discuss on the extension of the class of $M-$matrices named M_v-matrices, where B is an eventually nonnegative matrix. Finally, we study the Schur complement of the class of M_v-matrices and present some results.

1. INTRODUCTION

The term $M-$matrix (Minkowski matrix) was first used by Ostrowski [15, 16] in reference to the work of Minkowski [10, 11], who proved that the determinant of $A \in \mathbb{Z}^{n,n}$ is positive if all of its row sums are positive. We defined as $M-$matrices the matrices of the form $A = sI - B$, where B is entrywise nonnegative ($B \geq 0$) and $0 \leq \rho(B) \leq s$.

After the Perron-Frobenius Theory was established, Friedland [7] introduced the class of eventually nonnegative matrices as the $n \times n$ real matrices A for which exists an integer $k_0 > 0$ such that $A^k \geq 0$ for all $k \geq k_0$. Recall that a matrix A is said to be eventually positive if there exists an integer $k_0 > 0$ such that $A^k > 0$ (A^k is entrywise positive) for all $k \geq k_0$.

In 2006, Noutsos [12] extended the Perron-Frobenius Theory by introducing the definitions of the Perron-Frobenius property and the strong Perron-Frobenius property. Recall that a matrix $A \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property if its dominant eigenvalue $\lambda_1 > 0$ and the corresponding eigenvector $x^{(1)} \geq 0$. A matrix $A \in \mathbb{R}^{n,n}$ possesses the strong Perron-Frobenius property if its dominant eigenvalue $\lambda_1 > 0$ and $\lambda_1 > |\lambda_i|, i = 2, 3, \ldots n$ and the corresponding eigenvector $x^{(1)} > 0$ [12].

The term pseudo $M-$matrix, the matrix $A = sI - B$, where $0 < \rho(B) < s$ and B being an eventually positive matrix, was introduced in 2004 by Johnson and Tarazaga [8]. Afterward, the term M_v-matrix was introduced in 2006 by Olesky et al.[14] for matrices $A = sI - B$, where $0 \leq \rho(B) \leq s$ and B is eventually nonnegative matrix. Later on, in 2008, Elhashash and Szyld [5] studied the class of $EM-$matrices, where $0 < \rho(B) \leq s$ and B being an eventually nonnegative matrix.

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Key words and phrases. $M-$matrices; M_v-matrices; Schur complement; Perron-Frobenius theory.

From the definition, the class of $EM-$matrices is strictly subclass of M_v-matrices, since B in the class of M_v-matrices may be a nilpotent matrix.

Finally, the class of generalized $M-$matrices or $GM-$matrices contain matrices of the form $A = sI - B$, where $0 < \rho(B) \leq s$ and both B, B^T possess the Perron-Frobenius property. An extension of $M-$matrices are used in many fields such as mathematics (iterative methods, discretizations of differential operators), economics (gross substitutability, stability of a general equilibrium and Leontief's input-output analysis in economic systems), optimization, Markov chains in the field of probability theory and operation research like queuing theory, engineering (control theory) and also biology (population dynamics).

Let $A \in \mathbb{R}^{n,n}$ and suppose A_{11} is a nonsingular principal submatrix of A. The Schur complement of A_{11} in A [4], denoted by (A/A_{11}), is defined as follows: Let $\hat{A}$ be the matrix obtained from A by simultaneous permutation of rows and columns which puts A_{11} into the upper left corner of $\hat{A}$.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{1.1}$$

Then, the Schur complement of A_{11} in A is

$$(A/A_{11}) = A_{22} - A_{21}{A_{11}}^{-1}A_{12} \tag{1.2}$$

and the Schur's formula is

$$\det A = \det A_{11} \det(A/A_{11}).$$

The study of the Schur complement of $M-$matrices was introduced by Crabtree [2, 3] and Ky Fan [6].

2. $M-$MATRICES

First, we give the example of various class of $M-$matrices categorized by definition:

Example 2.1. *Let a matrix* $A_1 = sI - B_1$ *with*

$$B_1 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \ s > 4.$$

$\rho(B_1) = 4$ *and* B_1 *a nonnegative matrix. Suppose* $s = 5$, *we have that*

$$A_1 = 5I - B_1 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -1 & 3 \end{bmatrix} \text{ is an } M-\text{matrix.}$$

Example 2.2. *Let a matrix* $A_2 = sI - B_2$ *with*

$$B_2 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \ s \geq 3.4142.$$

$\rho(B_2) = 3.4142$ *and* B_2 *is eventually nonnegative* ($\forall k \geq 3$, $B_2^k \geq 0$). *For every* $s \geq 3.4142$, A_2 *is an* $EM-$*matrix.*

Example 2.3. *Let a matrix* $A_3 = sI - B_3$ *with*

$$B_3 = \begin{bmatrix} 2 & -0.5 & 1 \\ 3 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}, \ s \geq 6.4326.$$

$\rho(B_3) = 6.4326$ *and* B_3 *is eventually nonnegative* ($\forall k \geq 5,\ B_3^k \geq 0$). *For every* $s \geq 6.4326$, A_3 *is an* M_v-*matrix. Obviously it is also an* $EM-$*matrix.*

Example 2.4. *Let a matrix* $A_4 = sI - B_4$ *with*

$$B_4 = \begin{bmatrix} 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ s \geq 0.$$

$\rho(B_4) = 0$ *and* B_4 *is eventually nonnegative* ($\forall k \geq 3,\ B_4^k \geq 0$). *Remark that* B_4 *is a nilpotent matrix* ($\forall k \geq 5,\ B_4^k = 0$). *For every* $s \geq 0$, A_4 *is* M_v-*matrix, however it is not an* $EM-$*matrix.*

Example 2.5. *Let a matrix* $A_5 = sI - B_5$ *with*

$$B_5 = \begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 3 & 1 & -1 \\ 5 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix}, \ s > 5.$$

$\rho(B_5) = 5$, B_5 *and* B_5^T *possess the Perron-Frobenius property with Perron-Frobenius eigenpairs* $\left(5,\ (0.5714\ 0.5714\ 1\ 1)^T\right)$ *and* $\left(5,\ (1\ 1\ 0\ 0)^T\right)$ *but* B *is not eventually nonnegative.* A_5 *is a* $GM-$*matrix.*

Some properties of the nonsingular $M-$matrices are given by Plemmons [17] as follows:

Theorem 2.6. [17] *Let* $A \in \mathbb{R}^{n,n}$, $n \geq 2$. *Then each of the following conditions is equivalent to the statement:* A *is a nonsingular* $M-$*matrix.*

(1) $(A + D)^{-1} > 0$, *for each diagonal matrix* $D \geq 0$.
(2) $(A + \alpha I)^{-1} > 0$, $\forall \alpha \geq 0$.
(3) *The inverse of each principal submatrix of* A *is positive.*
(4) *The inverse of each principal submatrix of* A *of orders* $1, 2$ *and* n *is positive.*

This theorem gave necessary and sufficient conditions of a nonsingular matrix to be an $M-$matrix.

Theorem 2.7. *A matrix* A *is a nonsingular* $M-$*matrix iff* A *is a* $Z-$*matrix (the class of matrices with nonpositive offdiagonal entries and nonnegative the diagonal ones) and* $A^{-1} \geq 0$.

The following theorem is concentrated to irreducible nonsingular $M-$matrices.

Theorem 2.8. *A matrix* $A \in \mathbb{R}^{n,n}$ *is an irreducible and nonsingular* $M-$*matrix iff* A *is a* $Z-$*matrix and* $A^{-1} > 0$.

Johnson and Tarazaga [8] generalized the property above to the class of pseudo $M-$matrices as follows:

Theorem 2.9. [8] *If $A = sI - B$ is a pseudo $M-$matrix, then A^{-1} is an eventually positive matrix ($A^{-1} \overset{v}{>} 0$).*

Proof. Suppose that $A = sI - B$ is a pseudo $M-$matrix and let B an eventually positive matrix with $s > \rho(B)$. Let $\lambda = \rho(B)$ and associated right and left eigenvector $Bx = \lambda x$ and $y^T B = \lambda y^T$, respectively, with $x, y > 0$. Because $A = sI - B$, we can say that $(s - \lambda)^{-1}$ is a strictly dominant positive eigenvalue for A^{-1}.

Then, $Ax = (s-\lambda)x$ with x a right eigenvector of A^{-1} associated with $(s-\lambda)^{-1}$, and similarly, y^T is a left eigenvector of A^{-1}. Hence, from the Theorem 1 [8], A^{-1} is eventually positive. □

The study of nonsingular $M-$matrix that created with irreducible eventually nonnegative matrices (M_v-matrices) was extended by Le and McDonald [9] and then by Olesky et al. [14]. Some properties are presented here.

Theorem 2.10. [14] *Suppose an M_v-matrix $A \in \mathbb{R}^{n,n}$, $A = sI - B$ with B an eventually nonnegative matrix and $0 \le \rho(B) \le s$. Then*

(1) *$s - \rho(B) \in \sigma(A)$.*
(2) *$Re\lambda \ge 0$, $\forall \lambda \in \sigma(A)$.*
(3) *$\det(A) \ge 0$, $\det(A) = 0$ iff $s = \rho(B)$.*
(4) *If, in particular, $\rho(B) > 0$, then there exists an eigenvector $x \ge 0$ of A and an eigenvector $y \ge 0$ of A^T corresponding to $\lambda(A) = s - \rho(B)$.*
(5) *If, in particular, B is eventually positive and $s > \rho(B)$, then in (4) $x > 0, y > 0$ and in (2) $Re\lambda > 0$, $\forall \lambda \in \sigma(A)$.*

Theorem 2.11. [14] *Let $A \in \mathbb{R}^{n,n}$ written in a form $A = sI - B$, where B is an eventually nonnegative and B has a positive eigenvector corresponding to $\rho(B)$. Consider the following conditions:*

(1) *A is an M_v-matrix.*
(2) *There exists an invertible diagonal matrix $D \ge 0$ such that the row sums of AD are nonnegative.*
(3) *$\exists x > 0$ such that $Ax \ge 0$.*

Then, (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). If, in addition, B is not nilpotent, then all conditions (1) − (3) are equivalent.

We can give an example to see how Theorem 2.11 works.

Example 2.12. *Let a matrix $A = sI - B$ with*

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \ s \ge 2.5616 \ .$$

$\rho(B) = 2.5616$ and B is eventually nonnegative ($\forall k \ge 4$, $B^k \ge 0$).

Suppose $s = 3$, then $A = 3I - B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ *is an M_v-matrix.*

There exists an invertible diagonal matrix $D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ge 0$

such that $AD = \begin{bmatrix} 6 & -2 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ *and every row sum of AD is nonnegative. In addition,* $\exists x = (1\ 1\ 1\ 1)^T > 0$ *such that* $Ax = (1\ 1\ 2\ 0)^T \geq 0$.

The next theorem present some properties of singular M_v-matrices:

Theorem 2.13. [14] *Let a singular* M_v-*matrix* $A \in \mathbb{R}^{n,n}$ *written in the form* $A = sI - B$, *where B is eventually positive. Then, the following hold.*

(1) $rank(A) = n - 1$.
(2) $\exists x > 0$ *such that* $Ax = 0$.
(3) *If for some vector* $u \neq x$ *such that* $Au \geq 0$, *then* $u = 0$.

The properties of generalized $M-$matrices or $GM-$matrices were given by Elhashash and Szyld [5] as follows:

Theorem 2.14. [5] *Let a matrix* $A \in \mathbb{R}^{n,n}$ *whose eigenvalues have the order* $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. *Then, A is a nonsingular* $GM-$*matrix iff* A^{-1} *and* ${A^{-1}}^T$ *possess the Perron-Frobenius property and for the eigenvalues of A there hold* $0 < \lambda_n < Re(\lambda_i)$ *for all* $\lambda_i \neq \lambda_n$.

Corollary 2.15. [5] *A matrix* $A \in \mathbb{R}^{n,n}$ *is a nonsingular* $GM-$*matrix iff A and* A^T *possess the Perron-Frobenius property and* $Re(\lambda^{-1}) > \rho(A)^{-1}$, $\forall \lambda \in \sigma(A)$, $\lambda \neq \rho(A)$.

Corollary 2.16. [5] *Every real eigenvalue of a nonsingular* $GM-$*matrix is positive.*

In Example 2.5, A_5 is not a nonsingular $GM-$matrix because A_5^T does not possess the Perron-Frobenius property (the eigenpair of A_5^T is $\big(5,\ (-1\ -1\ 0.5714\ 0.5714)^T\big)$).

From the definition, the class of $M-$matrices is a subclass of $EM-$matrices and the class of pseudo $M-$matrices is also a subclass of $EM-$matrices, however an $M-$matrix may not be a pseudo $M-$matrix. The class of $EM-$matrices is a subclass of M_v-matrices and the class of M_v-matrices is also a subclass of $GM-$matrices because for every B eventually nonnegative, both B and B^T possess Perron-Frobenius property (see Theorem 2.3, [12]). The description given above is showed in the diagram below.

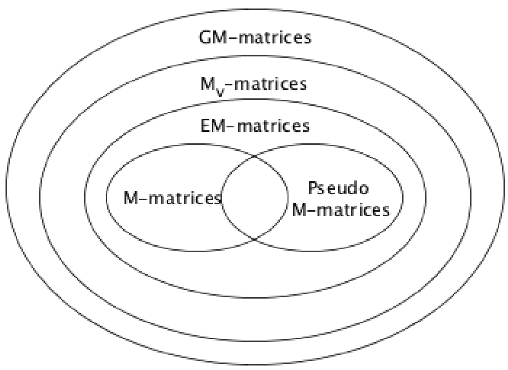

FIGURE 1. A diagram shows the relations between various class of $M-$matrices using the Perron-Frobenius property.

3. Schur complement of various class of $M-$matrices

Let $A = sI - B$, where B is entrywise nonnegative and $0 \leq \rho(B) \leq s$, an $M-$matrix of the form (1.1). Crabtree showed that the Schur complement of an $M-$matrix is also an $M-$matrix [2] and his work with Haynsworth [4] also tell us about the quotient property of the Schur complement of a matrix (see theorem in [4]). The associated B of the Schur complement (A/A_{11}), denoted by B_φ is

$$B_\varphi = sI - (A/A_{11})$$

and we can use the same s for B_φ, since $\rho(B_\varphi) \leq \rho(B) < s$ (see Theorem 2.7, Noutsos [12]).

Watford Jr. [19] studied the Schur complement of a $GM-$matrix with respect to the theory of cone in vector spaces and to the minimal eigenvector of a $GM-$matrix, we have result that the Schur complement of a $GM-$matrix is a $GM-$matrix if the matrix belongs in some direct sum cone. One question arises here: Is the Schur complement of an M_v-matrix also an M_v-matrix for every case or do we have to find some special conditions of M_v-matrices to guarantee the M_v- property of Schur complement?

From the matrices of examples 2.2, 2.3, 2.4 it can be checked that all their Schur complements are M_v-matrices for A_{11} being any principal submatrix of A. However, there is some M_v-matrix, whose Schur complement is not an M_v-matrix, as we can observe in the following example:

Example 3.1. *Let a matrix $A = sI - B$ with*

$$B = \begin{bmatrix} 3 & 3 & -1 & 1 \\ 3 & 3 & 1 & -1 \\ 5 & 3 & 1 & 1 \\ 3 & 5 & 1 & 1 \end{bmatrix}, \ s > 6.$$

$\rho(B) = 6$ and B is eventually nonnegative ($\forall k \geq 64$, $B^k \geq 0$).

Suppose $s = 8$, we have that $A = 8I - B = \begin{bmatrix} 5 & -3 & 1 & -1 \\ -3 & 5 & -1 & 1 \\ -5 & -3 & 7 & -1 \\ -3 & -5 & -1 & 7 \end{bmatrix}$ *is an M_v-matrix.*

Suppose $A_{11} = [5]$, $A_{22} = \begin{bmatrix} 5 & -1 & 1 \\ -3 & 7 & -1 \\ -5 & -1 & 7 \end{bmatrix}$, *then the Schur complement*

$$(A/A_{11}) = \begin{bmatrix} 3.2 & -0.4 & 0.4 \\ -6 & 8 & -2 \\ -6.8 & -0.4 & 6.4 \end{bmatrix}$$

is not an M_v-matrix because the associated B_φ is not an eventually nonnegative matrix (B_φ^T does not possess the Perron-Frobenius property).

The study of the Schur complement of an M_v-matrix can be separated into 2 cases: the irreducible and reducible M_v-matrices.

- *Reducible M_v-matrix*

Let $A = sI - B$ a reducible M_v-matrix and suppose A_{11} is a nonsingular principal submatrix of A. Suppose $\hat{A}$ be the matrix obtained from A by simultaneous permutation of rows and columns which puts A_{11} into the upper left corner of $\hat{A}$ then we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

Then, the Schur complement of A_{11} in A is

$$(A/A_{11}) = A_{22} - 0\ {A_{11}}^{-1} A_{12} = A_{22}.$$

If A_{22} is an M_v-matrix, then it is trivial that the Schur complement is also an M_v-matrix.

- *Irreducible M_v-matrix*

This case is still in process. We can give some observations as follows:

Let $A = sI - B$ an irreducible M_v-matrix of the form (1.1) and B written as:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

⋄ Suppose that both submatrix A_{11} and A_{22} of A are M_v-matrices, then the associated Schur complement (A/A_{11}) is also an M_v-matrix.

Example 3.2. *Let an M_v-matrix $A = sI - B$ with*

$$B = \begin{bmatrix} 1 & 2 & 3 & -1 & 1 & 1 \\ 3 & 1 & 2 & 1 & -1 & 1 \\ 2 & 3 & 1 & 1 & 1 & -1 \\ 5 & 4 & 7 & 1 & 1 & 1 \\ 7 & 5 & 4 & 1 & 1 & 1 \\ 4 & 7 & 5 & 1 & 1 & 1 \end{bmatrix}, \quad s > 8.772.$$

$\rho(B) = 8.772$ and B is an eventually nonnegative matrix.

Suppose $s = 11$, we have that $A = 11I - B = \begin{bmatrix} 10 & -2 & -3 & 1 & -1 & -1 \\ -3 & 10 & -2 & -1 & 1 & -1 \\ -2 & -3 & 10 & -1 & -1 & 1 \\ -5 & -4 & -7 & 10 & -1 & -1 \\ -7 & -5 & -4 & -1 & 10 & -1 \\ -4 & -7 & -5 & -1 & -1 & 10 \end{bmatrix}$ *is an M_v-matrix.*

Suppose $A_{11} = \begin{bmatrix} 10 & -2 & -3 & 1 \\ -3 & 10 & -2 & -1 \\ -2 & -3 & 10 & -1 \\ -5 & -4 & -7 & 10 \end{bmatrix}$ *and* $B_{11} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 1 \\ 5 & 4 & 7 & 1 \end{bmatrix}$ *an eventually nonnegative matrix,* $A_{22} = \begin{bmatrix} 10 & -1 \\ -1 & 10 \end{bmatrix}$ *and* $B_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *a positive matrix.*

Then, the Schur complement

$$(A/A_{11}) = \begin{bmatrix} 8.4021 & -2.6298 \\ -2.3745 & 8.4021 \end{bmatrix}$$

is an M_v-matrix because the associated B_φ is a positive matrix. (Obviously, it is an $M-$matrix.)

⋄ If A_{22} is an M_v-matrix, then the associated Schur complement (A/A_{11}) is always an M_v-matrix, even if A_{11} is not.
⋄ If A_{22} is not an M_v-matrix, then the associated Schur complement (A/A_{11}) may not be an M_v-matrix, or may be.

Here is an example for the Schur complement (A/A_{11}) which is not an M_v-matrix.

Example 3.3. *Let an M_v-matrix $A = sI - B$ with*

$$B = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & 1 & -1 \end{bmatrix}, \quad s > 4.5734.$$

$\rho(B) = 4.5734$ and B is an eventually nonnegative matrix.

Suppose $s = 7$, we have that $A = 7I - B = \begin{bmatrix} 6 & -3 & -2 & 1 \\ -1 & 5 & -1 & -3 \\ 0 & -1 & 5 & 1 \\ -1 & -2 & -1 & 8 \end{bmatrix}$ *is an M_v-matrix.*

Suppose $A_{11} = \begin{bmatrix} 6 & -3 \\ -1 & 5 \end{bmatrix}$ *and* $B_{11} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 5 & 1 \\ -1 & 8 \end{bmatrix}$ *and* $B_{22} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ *is not an eventually nonnegative matrix.*

Then, the Schur complement

$$(A/A_{11}) = \begin{bmatrix} 4.7037 & 0.3704 \\ -2.0741 & 6.5926 \end{bmatrix}$$

is not an M_v-matrix because the associated B_φ is not an eventually nonnegative matrix.

Here is an example for the Schur complement (A/A_{11}) which is also an M_v-matrix.

Example 3.4. *Let an M_v-matrix $A = sI - B$ with*

$$B = \begin{bmatrix} 10 & 3 & 2 & 0 & 1 & 4 & 8 & -1 \\ 3 & 0 & 4 & 1 & 2 & 5 & 1 & -4 \\ 2 & 4 & 3 & 6 & 8 & -2 & 1 & 4 \\ 3 & 4 & 8 & 3 & 1 & -1 & 4 & 2 \\ 2 & 1 & -1 & 2 & 3 & 4 & 2 & 4 \\ 1 & 2 & 3 & 1 & 2 & 4 & 3 & 1 \\ 3 & -1 & 5 & 7 & 8 & 4 & 2 & 0 \\ 23 & 3 & -14 & 5 & 4 & 3 & -2 & -10 \end{bmatrix}, \quad s > 22.0670.$$

$\rho(B) = 22.0670$ and B is an eventually nonnegative matrix.

Suppose $s = 24$, we have that
$$A = 24I - B = \begin{bmatrix} 14 & -3 & -2 & 0 & -1 & -4 & -8 & 1 \\ -3 & 24 & -4 & -1 & -2 & -5 & -1 & 4 \\ -2 & -4 & 21 & -6 & -8 & 2 & -1 & -4 \\ -3 & -4 & -8 & 21 & -1 & 1 & -4 & -2 \\ -2 & -1 & 1 & -2 & 21 & -4 & -2 & -4 \\ -1 & -2 & -3 & -1 & -2 & 20 & -3 & -1 \\ -3 & 1 & -5 & -7 & -8 & -4 & 22 & 0 \\ -23 & -3 & 14 & -5 & -4 & -3 & 2 & 34 \end{bmatrix}$$

is an M_v-matrix.

Suppose $A_{11} = \begin{bmatrix} 14 & -3 \\ -3 & 24 \end{bmatrix}$ *and* $B_{11} = \begin{bmatrix} 10 & 3 \\ 3 & 0 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 21 & -6 & -8 & 2 & -1 & -4 \\ -8 & 21 & -1 & 1 & -4 & -2 \\ 1 & -2 & 21 & -4 & -2 & -4 \\ -3 & -1 & -2 & 20 & -3 & -1 \\ -5 & -7 & -8 & -4 & 22 & 0 \\ 14 & -5 & -4 & -3 & 2 & 34 \end{bmatrix}$

and $B_{22} = \begin{bmatrix} 3 & 6 & 8 & -2 & 1 & 4 \\ 8 & 3 & 1 & -1 & 4 & 2 \\ -1 & 2 & 3 & 4 & 2 & 4 \\ 3 & 1 & 2 & 4 & 3 & 1 \\ 5 & 7 & 8 & 4 & 2 & 0 \\ -14 & 5 & 4 & 3 & -2 & -10 \end{bmatrix}$ *is not an eventually nonnegative matrix.*

Then, the Schur complement

$$(A/A_{11}) = \begin{bmatrix} 19.8746 & -6.1896 & -8.5627 & 0.3180 & -2.6575 & -3.0581 \\ -9.3089 & 20.8012 & -1.6544 & -1.0214 & -6.2538 & -0.9480 \\ 0.4434 & -2.0612 & 20.7217 & -4.9297 & -3.3089 & -3.5994 \\ -3.5627 & -1.0948 & -2.2813 & 19.1590 & -3.8287 & -0.5291 \\ -5.3609 & -6.9847 & -8.1804 & -4.7676 & 20.3272 & 0.1498 \\ 9.2110 & -5.3394 & -6.3945 & -11.5596 & -12.0642 & 37.0734 \end{bmatrix}$$

is an M_v-matrix because the associated B_φ is an eventually nonnegative matrix ($\forall k \geq 6$, $B^k \geq 0$).

In the last case we have to find particular conditions that guarantee the M_v-property of the Schur complement.

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VARIATIONAL SOLUTIONS OF BVPS VARIATIONAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR MONOTONE DISCRETE INCLUSIONS IN HILBERT SPACES

KONSTANTINA G. PALASKA

ABSTRACT. We use variational methods to investigate the existence and uniqueness of solutions of a two-point boundary value problem concerning a system of second order difference inclusions in a Hilbert space, when the operators involved are maximal monotone.

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MINIMAL HYPERSURFACES IN $\mathbb{R}^4$

THEODOROS KASIOUMIS

Abstract. We investigate complete minimal hypersurfaces $f : M^3 \to \mathbb{R}^4$, with Gauss-Kronecker curvature identically zero and nowhere vanishing second fundamental form. If the scalar curvature is bounded from below we prove that $f(M^3)$ splits as a Euclidean product $L^2 \times \mathbb{R}$, where L^2 is a complete minimal surface in $\mathbb{R}^4$ with Gaussian curvature bounded from below.

1. Some known definitions

Definition 1.1. *Manifold of dimension k is a Hausdorff topological space, whose topology has countable base, that is locally homeomorphic to $\mathbb{R}^k$, by a collection (called an atlas) of homeomorphisms called charts.*

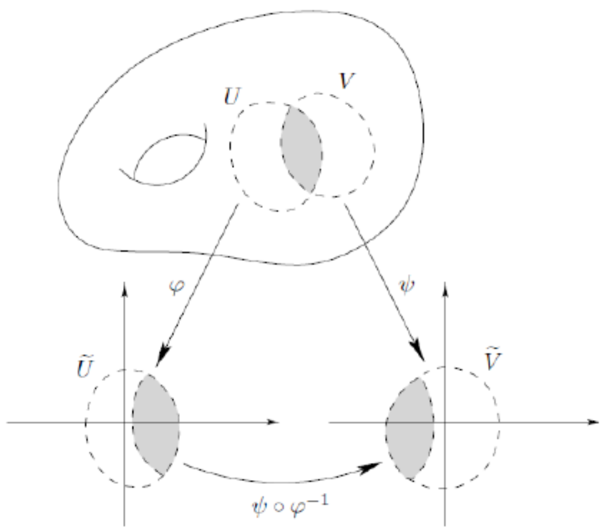

Figure 1

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Definition 1.2. *A differentiable map $f : M^n \to \tilde{M}^k$ is called* **immersion** *if at every $p \in M^n$ the differential $df_p : T_pM^n \to T_{f(p)}\tilde{M}^k$ is 1-1. If in addition f is a homeomorphism onto $f(M^n)$ then f is called* **embedding**.

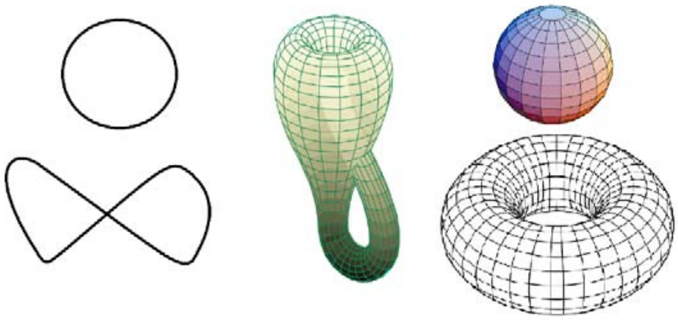

FIGURE 2

Definition 1.3. *An immersion $f : (M^n, \langle\cdot,\cdot\rangle) \to (\tilde{M}^k, \langle\cdot,\cdot\rangle)$ is called* **isometric immersion** *if $\langle X, Y\rangle_M = \langle df(X), df(Y)\rangle_{\tilde{M}}$ for $X, Y \in \mathfrak{X}(M^n)$.*

Given an isometric immersion f we can construct more by composing with an isometry of the ambient space. When we say that an isometric immersion is unique we mean modulo isometries of the ambient space.

Definition 1.4. *An isometric immersion $f : M^n \to \mathbb{Q}_c^k$ is called* **rigid** *if every other isometric immersion g is a composition with an isometry T of the ambient space, i.e, $g = T \circ f$.*

The second fundamental form $\alpha_f : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(N_fM)$ is a symmetric billinear form defined as the difference

$$\alpha_f(X, Y) = \big(\tilde{\nabla}_X Y\big) - \big(\nabla_X Y\big)$$

where $\tilde{\nabla}$ is the connection of the ambient space and ∇ the Levi-Civita connection of M.

The corresponding self-adjoint linear tranformation of tangent space A_ξ is called shape operator and $\langle \alpha_f(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle$ is the relationship between them.

Definition 1.5. *The $Gauss-Knonecker$ curvature is the determinant of the shape operator, $K = det(A_\xi)$ and its trace $H = trace(A_\xi)$ is called mean curvature.*

A hypersurface is called **Minimal** if $H \equiv 0$.

Definition 1.6. *The curvature tensor is $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

When the ambient space is a space of constant sectional curvature $\mathbb{Q}_c^n$ the curvature tensor becomes

$$R(X, Y)Z = c\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big).$$

If we pick an orthonormal frame $\{e_i\}, i = 1, \dots n$ on the tangent space of M we can define the following curvatures:

Definition 1.7.

$$Ric(X) = \sum_{i=1}^{n} \langle R(e_i, X)X, e_i \rangle$$

$$\tau(x) = \sum_{i=1}^{n} Ric(e_i)$$

2. MAIN RESULTS

Theorem 2.1. (Dajczer-Gromoll (1985)) *Let $g : M^{n-k} \to \mathbb{S}^n$, $n \geq 4$, be an isometric immersion and $\gamma \in C^\infty(M)$. Then the map $\Psi : N_g M \to \mathbb{R}^n$ defined as*

$$\Psi(x, w) = \gamma(x) g(x) + dg(\nabla\gamma) + w,$$

on the open subest of regular points, is an immersed hypersurface of $\mathbb{R}^{n+1}$ with constant index of relative nullity k. Conversely, any hypersurface of $\mathbb{R}^{n+1}$ with constant index of relative nullity k can be parametrized this way, at least locally.

Theorem 2.2. (Dajczer - Gromoll (1985)*) If $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, is a minimal isometric immersion of a complete Riemannian manifold M^n, then any other minimal isometric immersion $g : M^n \to \mathbb{R}^{n+1}$ is congruent to f, i.e $g = T \circ f$ where T is an isometry of $\mathbb{R}^{n+1}$, unless M^n splits as a Riemannian product $M^n = L^3 \times \mathbb{R}^{n-3}$.*

Question (T. Hasanis-A. S. Halilaj-T. Vlachos):

Is it true that any complete minimal hypersurface with vanishing Gauss-Kronecker curvature in $\mathbb{R}^4$ is a cylinder over a minimal surface in $\mathbb{R}^3$?

Before answering to the question, let us give an example on how to construct minimal surfaces with vanishing Gauss-Kronecker curvature in $\mathbb{R}^4$.

Examples of Minimal surfaces in $\mathbb{R}^4$

Let $g : M^2 \to \mathbb{R}^3$ be a complete minimal surface and $i : \mathbb{R}^3 \to \mathbb{R}^4$ the inclusion map. Then the cylinder $f : M^2 \times \mathbb{R} \to \mathbb{R}^4$, $f(x, t) = (i \circ g)(x) + te_4$ is a complete minimal hypersurface in $\mathbb{R}^4$.

Theorem 2.3. (T. Hasanis - A. S. Halilaj - T. Vlachos)(2005) *The answer is positive under the assumption that the second fundamental form is nowhere vanishing and the scalar curvature bounded from below.*

An Idea of the Proof: Without loss of generality we may assume that M^3 is simply connected, after passing to the universal covering space. The standard monodromy argument allows us to define a global orthonormal frame field $\{e_1, e_2, e_3\}$ of principal directions. Consider the functions $u = \langle \nabla_{e_3} e_1, e_2 \rangle$, $v = e_2(log\lambda)$, which will play a crucial role in the proof of the Theorem. Using the structural equations :

$$\nabla_X Y - \nabla_Y X = [X, Y],$$
$$[\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]} Z = R(X, Y)Z,$$

see *T. Hasanis- A. S. Halilaj-T. Vlachos* and the Codazzi equation

$$(\nabla_X A_\xi)Y - A_{\nabla^\perp_X \xi} Y = (\nabla_Y A_\xi)X - A_{\nabla^\perp_Y \xi} X,$$

we derive the following system :

$$e_1(u) = e_3(v), \quad e_3(u) = -e_1(v)$$

$$e_2(u) = 2uv, \quad e_2(v) = v^2 - u^2$$
$$\langle \nabla_{e_2} e_1, e_2 \rangle = 0, \quad \langle \nabla_{e_2} e_2, e_3 \rangle = 0$$
$$[e_1, e_3] = -\frac{1}{2} e_3(log\lambda) e_1 - 2ue_2 + \frac{1}{2} e_1(log\lambda) e_3.$$

Calculating the Laplacian of u, v we deduce that u, v are harmonic functions. Therefore

$$\Delta(u^2 + v^2) \geq 2(u^2 + v^2).$$

Furthermore, the *Ricci* curvature is bounded from below since by assumption the scalar curvature is bounded from below. Now, using a result due to S. Y. Cheng, S. T. Yau, we deduce that

$$sup(u^2 + v^2) = 0.$$

Consequently, λ is constant along the integral curves of e_2. Consider the 2-dimensional distribution V which is spanned by e_1 and e_3. Because $u \equiv 0$, we see that V is involutive.

A result by *T. Hasanis - A. S. Halilaj-T. Vlachos (2005).*

Let L_x^2 be a maximal integral submanifold of V passing through a point x of M^3, and denote by $i : L_x^2 \to M^3$ its inclusion map. Then $\tilde{f} = f \circ i : L_x^2 \to \mathbb{R}^4$ defines and minimal surface with bounded *Gaussian* curvature which lies in $\mathbb{R}^3$ and $df(e_2)$ is constant along $\tilde{f}$. Hence $f(M^3)$ is a cylinder over a minimal surface of $\mathbb{R}^3$.

Question:

Can we construct a non totally geodesic, complete, minimal hypersurface with bounded $Gauss - Kronecker$ curvature exist in $\mathbb{R}^4$?

Question:

Can we construct a complete minimal hypersurface in $\mathbb{R}^4$ with $Gauss{-}Kronecker$ curvature identically zero and scalar curvature not bounded from below ?

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DIFFERENCE AND ALGEBRAIC EQUATIONS

LAZAROS MOYSIS AND NICHOLAS KARAMPETAKIS

Abstract. Systems of linear difference and algebraic equations $A(\sigma)\beta(k) = 0$, where σ denotes the shift forward operator and $A(\sigma)$ a regular polynomial matrix, give rise to both forward and backward propagating solutions. In the present work the problem of constructing such a system is studied. That is, given a set of discrete time forward and backward propagating functions $\beta(k)$, a method for constructing a family of matrices $A(\sigma)$ is proposed, such that the system $A(\sigma)\beta(k) = 0$ has the exact prescribed solutions.

1. Introduction

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[\sigma]$ the ring of polynomials with coefficients from $\mathbb{R}$ and $\mathbb{R}(\sigma)$ the field of rational functions. By $\mathbb{R}^{p\times m}[\sigma]$, $\mathbb{R}^{p\times m}(s)$, $\mathbb{R}^{p\times m}_{pr}(\sigma)$ we denote the sets of $p \times m$ polynomial, rational and proper rational matrices respectively, with real coefficients. We are going to study the behavior of systems of algebraic and difference equations that are described in the form of an *(Auto-Regressive) AR-representation*, that is

$$A(\sigma)\beta(k) = 0 \tag{1.1}$$

with $k = 0, 1, ..., N - q$, or equivalently

$$A_q\beta(k + q) + A_{q-1}\beta(k + q - 1) + ... + A_0\beta(k) = 0 \tag{1.2}$$

where $\beta(k) \in \mathbb{R}^r$ is the state of the system, σ denotes the forward shift operator $\sigma\beta(k) = \beta(k + 1)$ and

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + ... + A_1\sigma + A_0 \in \mathbb{R}^{r\times r}[\sigma] \tag{1.3}$$

is a regular polynomial matrix with $\det[A(\sigma)] \neq 0$ and $A_q \neq 0$. The number q is often called the *lag* of the matrix.

Such systems often appear in system theory, since they accurately model many economic, biological and other discrete time phenomena. For example, the Leslie Population Growth Model in biology and the Leontief Model of a multisector economy in economics [5] are both examples of singular systems, which are easily seen to be a special case of AR systems (1.1).

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The solution space of such systems consists of both forward and backward solutions and is denoted as

$$B := \{\beta(k) : [0, N] \to \mathbb{R}^r \mid (1.1) \text{ is satisfied } \forall k \in [0, N-q]\,\} \tag{1.4}$$

The forward solution space, i.e. the vector space that consists of solutions $\beta(k)$ propagating forward in time (with given initial conditions) is connected to the finite elementary divisor structure of $A(\sigma)$. The backward solution space is the vector space consisting of solutions $\beta(k)$ propagating backward in time, (with given final conditions), and it is connected to the infinite elementary divisor structure of $A(\sigma)$. The algebraic structure of polynomial matrices has been studied in [3, 6, 7, 8, 9, 10, 11, 14, 18]. Symbolic and numerical algorithms have also been developed for the computation of the Jordan chains and the Smith form of polynomial matrices in [20] and [21].

The construction of the solution space of such systems has been previously studied by various authors, initially in [8] and later in [1] whereas an extension of the method in [8] to non regular systems is given in [12]. In this paper, we study the inverse problem, that is: Given a certain forward/backward solution space, find a system of algebraic/difference equations with the prescribed solution space. A partial solution to this problem has been described in [8], where only the smooth behavior for continuous time and the forward behavior for discrete time systems was studied. This method was later extended in [13] for continuous time systems, to include both the smooth and impulsive behavior and in [15] for discrete time systems to include both the forward and backward behavior. Both these methods though rely on the computation of the Jordan Pairs of $A(\sigma)$ and cannot be extented to non-regular systems with the current theory. In addition, they are much less versatile in handling the free parameters of the matrices $A_0, ..., A_q$ and require a deep understanding of the structure of polynomial matrices.

In this paper, we shall further extend these results for the case where both a forward and backward behavior is under consideration by using a novel methodology. The core of our proposed method lies in the fact that the vectors that consist a solution of the system (forward or backward), actually satisfy a certain system of equations, which we are going to solve in terms of the unknown coefficients of A(σ), in order to receive the original system. In each section examples are given to showcase the results.

It should also be noted that the problem of constructing a system with prescribed solutions has been studied in the field of behavioral theory by [2, 22, 23, 24, 25, 26], although the approach used is completely different.

2. Preliminaries

In this section we provide some background regarding the algebraic structure of polynomial matrices.

Definition 2.1. [19] A square polynomial matrix $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ is called *unimodular* if $\det A(\sigma) = c \in \mathbb{R}$, $c \neq 0$. A rational matrix $A(\sigma) \in \mathbb{R}_{pr}^{r\times r}(\sigma)$ is called *biproper* if $\lim_{\sigma\to\infty} A(\sigma) = E \in \mathbb{R}^{r\times r}$ with $rankE_{\mathbb{R}} = r$.

Theorem 2.1. [19] *Let $A(\sigma)$ as in (1.3). There exist unimodular matrices $U_L(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$,$U_R(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ such that*

$$S^{\mathbb{C}}_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = diag\left(1, ..., 1, f_z(\sigma), f_{z+1}(\sigma), ..., f_r(\sigma)\right) \tag{2.1}$$

with $1 \leq z \leq r$ *and* $f_j(\sigma)/f_{j+1}(\sigma)$ $j = z, z+1, ..., r$. $S^{\mathbb{C}}_{A(\sigma)}(\sigma)$ *is called the* Smith form *of* $A(\sigma)$ *(in* $\mathbb{C}$*) where* $f_j(\sigma) \in \mathbb{R}[\sigma]$ *are the* invariant polynomials *of* $A(\sigma)$. *The zeros* $\lambda_i \in \mathbb{C}$ *of* $f_j(\sigma)$, $j = z, z+1, ..., r$ *are called* finite zeros *of* $A(\sigma)$. *Assume that* $A(\sigma)$ *has* ℓ *distinct zeros. The partial multiplicities* $n_{i,j}$ *of each zero* $\lambda_i \in \mathbb{C}$, $i = 1, ..., \ell$ *satisfy*

$$0 \leq n_{i,z} \leq n_{i,z+1} \leq ... \leq n_{i,r} \tag{2.2}$$

with

$$f_j(\sigma) = (\sigma - \lambda_i)^{n_{i,j}} \hat{f}_j(\sigma) \tag{2.3}$$

$j = z, ..., r$ *and* $\hat{f}_j(\lambda_i) \neq 0$ *The terms* $(\sigma - \lambda_i)^{n_{i,j}}$ *are called* finite elementary divisors *of* $A(\sigma)$ *at* λ_i.

Denote by n *the sum of the degrees of the finite elementary divisors of* $A(\sigma)$, *i.e.*

$$n := \deg\left[\prod_{j=z}^{r} f_j(\sigma)\right] = \sum_{i=1}^{\ell}\sum_{j=z}^{r} n_{i,j} \tag{2.4}$$

Similarly, we can find $U_L(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$, $U_R(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$ *having no poles and zeros at* $\sigma = \lambda_0$ *such that*

$$S^{\lambda_0}_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = diag\left(1, ..., 1, (\sigma - \lambda_0)^{n_z}, ..., (\sigma - \lambda_0)^{n_r}\right) \tag{2.5}$$

$S^{\lambda_0}_{A(\sigma)}(\sigma)$ *is called the Smith form of* $A(\sigma)$ *at the local point* λ_0.

Theorem 2.2. [19] *Let* $A(\sigma)$ *defined in* (1.3). *There exist biproper matrices* $U_L(\sigma) \in \mathbb{R}^{r\times r}_{pr}(\sigma)$, $U_R(\sigma) \in \mathbb{R}^{r\times r}_{pr}(\sigma)$ *such that*

$$U_L(\sigma)A(\sigma)U_R(\sigma) = S^{\infty}_{\mathrm{A}(\sigma)}(\sigma) = diag\left(\underbrace{\sigma^{q_1}, \sigma^{q_2}, ..., \sigma^{q_k}}_{k}, \overbrace{\frac{1}{\sigma^{\hat{q}_{k+1}}}, \frac{1}{\sigma^{\hat{q}_{k+2}}}, ..., \frac{1}{\sigma^{\hat{q}_r}}}^{r-k}\right) \tag{2.6}$$

with $1 \leq k \leq r$, $q_1 \geq \ldots \geq q_k > 0$ *and* $\hat{q}_r \geq \hat{q}_{r-1} \geq \ldots \hat{q}_{k+1} > 0$. $S^{\infty}_{\mathrm{A}(\sigma)}(\sigma)$ *is called the* Smith form of $A(\sigma)$ at infinity. *The first* k *terms* $\sigma^{q_1}, ..., \sigma^{q_k}$ *(resp. the latter* $(r-k)$ *terms* $\sigma^{\hat{q}_{k+1}}, ..., \sigma^{\hat{q}_r}$*) are the* poles *(resp.* zeros*)* at $\sigma = \infty$ *of* $A(\sigma)$. *It is proved in* [19] *that* $q_1 = q$.

Definition 2.2. [19] The dual polynomial matrix of $A(\sigma)$ is defined as

$$\tilde{A}(\sigma) := \sigma^q A(\frac{1}{\sigma}) = A_0\sigma^q + A_1\sigma^{q-1} + ... + A_q \tag{2.7}$$

Theorem 2.3. [19] *Let* $\tilde{A}(\sigma)$ *as in* (2.7). *There exist matrices* $\tilde{U}_L(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$, $\tilde{U}_R(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$ *having no poles or zeros at* $\sigma = 0$, *such that*

$$S^{0}_{\tilde{A}(\sigma)}(\sigma) = \tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = diag\left[\sigma^{\mu_1}, \ldots, \sigma^{\mu_r}\right] \tag{2.8}$$

$S^{0}_{\tilde{A}(\sigma)}(\sigma)$ *is the local Smith form of* $\tilde{A}(\sigma)$ *at* $\sigma = 0$. *The terms* σ^{μ_j} *are the finite elementary divisors of* $\tilde{A}(\sigma)$ *at zero and are called the* infinite elementary divisors (i.e.d.) *of* $A(\sigma)$.

The connection between the Smith form at infinity of $A(\sigma)$ and the Smith form at zero of the dual matrix is given in [10, 19]:

$$S^0_{\tilde{A}(\sigma)}(\sigma) = diag\left[1, \underbrace{\sigma^{q-q_2}, \ldots, \sigma^{q-q_k}}_{i.p.e.d.}, \underbrace{\sigma^{q+\hat{q}_{k+1}}, \ldots, \sigma^{q+\hat{q}_r}}_{i.z.e.d.}\right] \equiv diag\left[\sigma^{\mu_1}, \sigma^{\mu_2}, \ldots, \sigma^{\mu_r}\right] \tag{2.9}$$

where by i.p.e.d. and i.z.e.d. we denote the infinite pole and infinite zero elementary divisors respectively. From the above formula it is seen that the the orders of the infinite elementary divisors of $A(\sigma)$ are given by

$$\mu_1 = q - q_1 \overset{q=q_1}{=} 0 \tag{2.10a}$$

$$\mu_j = q - q_j \qquad j = 2, 3, ..., k \tag{2.10b}$$

$$\mu_j = q + \hat{q}_j \qquad j = k+1, ..., r \tag{2.10c}$$

We denote by μ the sum of the degrees of the infinite elementary divisors of $A(\sigma)$ i.e.

$$\mu := \sum_{j=1}^{r} \mu_j \tag{2.11}$$

Lemma 2.4. [1, 8, 19] *Let $A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + ... + A_0 \in \mathbb{R}^{r\times r}[\sigma]$. Let also n, μ be the sum of degrees of the finite and infinite elementary divisors of $A(\sigma)$, as defined previously. Then*

$$n + \mu = r \times q \tag{2.12}$$

The above relation is of fundamental importance in the sequel, since it connects the dimension of the forward and backward behavior (n and μ respectively) of the AR-representation (1.1) with the lag (q) and dimension (r) of $A(\sigma)$.

3. Modeling the forward behavior of a system described by an AR-Reresentation

In this section we present the forward behavior of (1.1) and propose a novel method of constructing a system that satisfies a prescribed forward behavior.

3.1. **Finite Elementary Divisors and Forward Solution Space.** Let us assume that $A(\sigma)$ has ℓ distinct zeros $\lambda_1, \lambda_2, ..., \lambda_\ell$ where for simplicity of notation we assume that $\lambda_i \in \mathbb{C}$, $i = 1, 2, ..., \ell$ and let $S^{\mathbb{C}}_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = blockdiag\left[1, ..., 1, f_z(\sigma), f_{z+1}(\sigma), ..., f_r(\sigma)\right]$. Assume that the partial multiplicities of the zeros $\lambda_i \in \mathbb{C}$ are $0 \leq n_{i,z} \leq n_{i,z+1} \leq ... \leq n_{i,r}$ i.e. $f_j(\sigma) = (\sigma - \lambda_i)^{n_{i,j}}\hat{f}_j(\sigma)$ $j = z, z+1, ..., r$ with $\hat{f}_j(\lambda_i) \neq 0$. Let $u_j(\sigma) \in \mathbb{R}[\sigma]^{r\times 1}$, be the columns of $U_R(\sigma)$ and $u_j^{(\varphi)}(\sigma) := (\partial^\varphi/\partial\sigma^\varphi)u_j(\sigma)$. Let also

$$\beta^i_{j,\varphi} := \tfrac{1}{\varphi!}u_j^{(\varphi)}(\lambda_i) \quad i = 1, 2, ..., \ell \quad j = z, z+1, \ldots, r \quad \varphi = 0, 1, ..., n_{i,j} - 1 \tag{3.1}$$

Define the vector valued functions

$$\beta^i_{j,\varphi}(k) := \lambda_i^k\beta^i_{j,\varphi} + k\lambda_i^{k-1}\beta^i_{j,\varphi-1} + ... + \binom{k}{\varphi}\lambda_i^{k-\varphi}\beta^i_{j,0} \quad \text{for } \lambda_i \neq 0 \tag{3.2a}$$

$$\beta^i_{j,\varphi}(k) := \delta(k)\beta^i_{j,\varphi} + \delta(k-1)\beta^i_{j,\varphi-1} + ... + \delta(k-\varphi)\beta^i_{j,0} \quad \text{for } \lambda_i = 0 \tag{3.2b}$$

where $i = 1, 2, ..., \ell$, $j = z, z+1, \ldots, r$, $\varphi = 0, 1, ..., n_{i,j} - 1$ and $\delta(k)$ or δ_k denotes the known Kronecker delta function.

Theorem 3.1. [12] *The vector valued functions* $\beta^i_{j,\phi}(k)$*, as defined in* (3.2)*, are solutions of* (1.1)*. In addition, let*

$$C_{i,j} := \begin{bmatrix} \beta^i_{j,0} & \beta^i_{j,1} & \dots & \beta^i_{j,n_{i,j}-2} & \beta^i_{j,n_{i,j}-1} \end{bmatrix} \tag{3.3}$$

$$J_{i,j} := \begin{bmatrix} \lambda_i & 1 & \dots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_{i,j} \times n_{i,j}} \tag{3.4}$$

where $i = 1, 2, ..., \ell$*,* $j = z, z+1, \dots, r$ *and*

$$C^F_i := \begin{bmatrix} C_{i,z} & C_{i,z+1} & \cdots & C_{i,r} \end{bmatrix} \quad ; \quad J^F_i := blockdiag \begin{bmatrix} J_{i,z} & \cdots & J_{i,r} \end{bmatrix} \tag{3.5}$$

Finally let

$$C^D_F := \begin{bmatrix} C^F_1 & \cdots & C^F_\ell \end{bmatrix} \quad ; \quad J^D_F := blockdiag \begin{bmatrix} J^F_1 & \cdots & J^F_\ell \end{bmatrix} \tag{3.6}$$

The forward solution space of the system (1.1) *is spanned by the columns of:*

$$B^D_F = \left\langle C^D_F \left(J^D_F\right)^k \right\rangle \tag{3.7}$$

and has dimension $\dim B^D_F = n$*, where* n *is defined in* (2.4)*.*

Theorem 3.2. *The vector valued functions* $\beta^i_{j,\varphi}(k)$ *defined in* (3.2)*, are solutions of* (1.1) *iff the vectors* $\beta^i_{j,\varphi}$ *satisfy the following system of equations:*

$$\begin{gathered} A(\lambda_i)\beta^i_{j,0} = 0 \\ A^{(1)}(\lambda_i)\beta^i_{j,0} + A(\lambda_i)\beta^i_{j,1} = 0 \\ \vdots \\ \frac{1}{(n_{i,j}-1)!} A^{(n_{i,j}-1)}(\lambda_i)\beta^i_{j,0} + \dots + A(\lambda_i)\beta^i_{j,(n_{i,j}-1)} = 0 \end{gathered} \tag{3.8}$$

Proof. We show that $\beta^i_{j,\varphi}(k)$ are solutions of (1.1) iff (3.8) are satisfied. First, consider the general case where $\lambda_i \neq 0$. For $\beta^i_{j,0}(k) = \lambda^k_i \beta^i_{j,0}$, we have:

$$A(\sigma)\lambda^k_i \beta^i_{j,0} = 0 \Leftrightarrow \tag{3.9}$$

$$A_q \lambda^{k+q}_i \beta^i_{j,0} + ... + A_1 \lambda^{k+1}_i \beta^i_{j,0} + A_0 \lambda^k_i \beta^i_{j,0} = 0 \Leftrightarrow \tag{3.10}$$

$$(A_q \lambda^q_i + ... + A_1 \lambda_i + A_0)\lambda^k_i \beta^i_{j,0} = 0 \Leftrightarrow \tag{3.11}$$

$$A(\lambda_i)\beta^i_{j,0} = 0 \tag{3.12}$$

so the first equation in (3.8) is proven. Now, letting $\beta^i_{j,1}(k) = k\lambda^{k-1}_i \beta^i_{j,0} + \lambda^k_i \beta^i_{j,1}$, we obtain:

$$\begin{gathered} A(\sigma)(k\lambda^{k-1}_i \beta^i_{j,0} + \lambda^k_i \beta^i_{j,1}) = 0 \Leftrightarrow \\ \left(A_q(k+q)\lambda^{k+q-1}_i \beta^i_{j,0} + ... + A_0 k \lambda^{k-1}_i \beta^i_{j,0}\right) + \left(A_q \lambda^{k+q}_i \beta^i_{j,1} + ... + A_0 \lambda^k_i \beta^i_{j,1}\right) = 0 \Leftrightarrow \\ \left(qA_q \lambda^{q-1}_i + ... + A_1\right) \lambda^k_i \beta^i_{j,0} + (A_q \lambda^q_i + ... + A_1 \lambda_i + A_0) k \lambda^k_i \beta^i_{j,0} + \\ +(A_q \lambda^q_i + ... + A_1 \lambda_i + A_0)\lambda^k_i \beta^i_{j,1} = 0 \end{gathered}$$

and taking into account that $A(\lambda_i)\beta^i_{j,0} = 0$ and (3.11), the above equation is written as

$$\left(qA_q \lambda^{q-1}_i + ... + A_1\right) \lambda^k_i \beta^i_{j,0} + (A_q \lambda^q_i + ... + A_1 \lambda_i + A_0)\lambda^k_i \beta^i_{j,1} = 0 \Leftrightarrow$$

$$\left(\left(qA_q\lambda_i^{q-1}+...+A_1\right)\beta^i_{j,0}+(A_q\lambda_i^q+...+A_1\lambda_i+A_0)\beta^i_{j,1}\right)\lambda_i^k=0\Leftrightarrow$$
$$A^{(1)}(\lambda_i)\beta^i_{j,0}+A(\lambda_i)\beta^i_{j,1}=0 \tag{3.13}$$

so the second equation in (3.8) holds true. Continuing inductively in the same fashion, the rest of the equations in (3.8) can be proven.

For the case where $\lambda_i=0$, letting $\beta^i_{j,0}(k)=\delta(k)\beta^i_{j,0}$ we obtain:

$$A(\sigma)\delta(k)\beta^i_{j,0}=0\Leftrightarrow$$
$$A_q\delta(k+q)\beta^i_{j,0}+...+A_1\delta(k+1)\beta^i_{j,0}+A_0\delta(k)\beta^i_{j,0}=0$$

and for $k=0$ this equation becomes $A_0\beta^i_{j,0}=A(0)\beta^i_{j,0}=0$. Now, letting $\beta^i_{j,1}(k)=\delta(k)\beta^i_{j,1}+\delta(k-1)\beta^i_{j,0}$ we obtain:

$$A(\sigma)(\delta(k)\beta^i_{j,1}+\delta(k-1)\beta^i_{j,0})=0\Leftrightarrow$$
$$A_q\delta(k+q)\beta^i_{j,1}+...+A_0\delta(k)\beta^i_{j,1}+A_q\delta(k+q-1)\beta^i_{j,0}+...+A_0\delta(k-1)\beta^i_{j,0}=0$$

which for $k=0$ yields $A_1\beta^i_{j,0}+A_0\beta^i_{j,1}=A^{(1)}(0)\beta^i_{j,0}+A(0)\beta^i_{j,1}=0$. Continuing in the same fashion, the rest of the equations in (3.8) can be proven for $\lambda_i=0$. □

The system of equations (3.8) can be rewritten in matrix form as

$$\begin{pmatrix}\frac{A^{(n_{i,j}-1)}(\lambda_i)}{(n_{i,j}-1)!} & \cdots & A(\lambda_i)\end{pmatrix}\underbrace{\begin{pmatrix}\beta^i_{j,0} & \cdots & 0\\ \vdots & \ddots & \vdots\\ \beta^i_{j,(n_{i,j}-1)} & \cdots & \beta^i_{j,0}\end{pmatrix}}_{W_{i,j}}=0_{r\times n_{i,j}} \tag{3.14}$$

for $i=1,2,...,\ell$, $j=z,z+1,\dots,r$ and $W_{i,j}\in\mathbb{R}^{rn_{i,j}\times n_{i,j}}$.

3.2. **Construction of a system with given forward behavior.** Theorem 3.2 is important, since it states that in order for a time sequence (3.2) to be a solution of $A(\sigma)\beta(k)=0$, the vectors $\beta^i_{j,\varphi}$ need to satisfy (3.14). Solving the above system of equations by calculating the left kernel of $W_{i,j}$, we can obtain the matrices, $A^{(n_{i,j}-1)}(\lambda_i)$ ..., $A'(\lambda_i)$, $A(\lambda_i)$, that represent the values of $A(\sigma)$ and its derivatives at λ_i. Thus the evaluation of $A(\sigma)$ is reduced to a Hermite interpolation problem. Alternatively, using the relation

$$\frac{A^{(\varepsilon)}(\lambda_i)}{\varepsilon!}=\binom{q}{\varepsilon}A_q{\lambda_i}^{q-\varepsilon}+\cdots+\binom{\varepsilon+1}{\varepsilon}A_{\varepsilon+1}\lambda_i+\binom{\varepsilon}{\varepsilon}A_\varepsilon\Rightarrow$$
$$\frac{A^{(\varepsilon)}(\lambda_i)}{\varepsilon!}=\begin{pmatrix}A_q & \cdots & A_\varepsilon\end{pmatrix}\begin{pmatrix}\binom{q}{\varepsilon}{\lambda_i}^{q-\varepsilon}I_r\\ \vdots\\ I_r\end{pmatrix}$$

for $\varepsilon=0,...,n_{i,j}-1$ we rewrite (3.14) as follows:

$$\begin{pmatrix}A_q & \cdots & A_0\end{pmatrix}Q_{i,j}W_{i,j}=0_{r\times n_{i,j}} \tag{3.15}$$

where

$$Q_{i,j} = \begin{pmatrix} \binom{q}{n_{i,j}-1}\lambda_i{}^{q-(n_{i,j}-1)} I_r & \cdots & \lambda_i{}^q I_r \\ \vdots & \ddots & \vdots \\ \binom{n_{i,j}}{n_{i,j}-1}\lambda_i I_r & \binom{r_{idx}}{c_{idx}}\lambda_i{}^{r_{idx}-c_{idx}} I_r & \vdots \\ I_r & \ddots & \lambda_i{}^2 I_r \\ \vdots & \ddots & \lambda_i I_r \\ 0 & \cdots & I_r \end{pmatrix} \in \mathbb{R}^{r(q+1)\times rn_{i,j}} \tag{3.16}$$

with $i = 1, 2, ..., \ell$, $j = z, z+1, \ldots, r$ and $r_{idx} = 0, ..., q$, $c_{idx} = 0, ..., (n_{i,j} - 1)$ the row and column indexes, counting from right to left and bottom to top.

In case where $n_{i,j} > q$, the derivatives of $A(\sigma)$ of order higher than q in (3.14) will be equal to zero. In this case, the matrices $Q_{i,j}, W_{i,j}$ in (3.15) take the following simplified form

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} \begin{pmatrix} I_r & \cdots & \lambda_i^q I_r \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_r \end{pmatrix} \begin{pmatrix} \beta^i_{j,n_{i,j}-q-1} & \cdots & \beta^i_{j,0} & \\ \vdots & & \ddots & \\ \beta^i_{j,n_{i,j}-1} & \cdots & \cdots & \beta^i_{j,0} \end{pmatrix} = 0_{r\times n_{i,j}} \tag{3.17}$$

Thus, our problem has been reduced to a linear system equation problem. That is, given a time sequence in the form of $\beta^i_{j,\varphi}(k)$, we can solve (3.15) in terms of the unknowns $A_0, A_1, ..., A_q$ in order to construct $A(\sigma)$. For the solution of such linear systems, numerous numerical methods exist. The most commonly used are *Singular Value Decomposition (SVD)* and the *QR Decomposition* [17]. These results give rise to the following algorithm for the construction of a system that satisfies a desired forward behavior.

Algorithm 3.1. *Suppose that a finite number of ℓ vector functions of the form*

$$\beta^F_{j,i}(k) = \lambda_i^k \beta^i_{j,n_{i,j}-1} + \cdots + \binom{k}{n_{i,j}-1}\lambda_i^{k-(n_{i,j}-1)}\beta^i_{j,0} \tag{3.18}$$

$$\beta^F_{j,i}(k) = \delta(k)\beta_{j,n_{i,j}-1} + ... + \delta\left(k - (n_{i,j}-1)\right)\beta_{j,0} \tag{3.19}$$

are given, with $i = 1, 2, ..., \ell \quad j = z, z+1, \ldots, r$.

Step 1: *Define $n = \sum_{i=1}^{\ell}\sum_{j=z}^{r} n_{ij}$.*

If r/n, then

$$q = \frac{n}{r}$$

else

$$q = [\frac{n}{r}] + 1$$

end if

Step 2: *Construct the matrices $Q_{i,j}, W_{i,j}$, defined in (3.14) and (3.16).*

Step 3: *Construct the combined matrices*

$$\begin{matrix} Q_i = \begin{pmatrix} Q_{i,z} & \cdots & Q_{i,r} \end{pmatrix} \in \mathbb{R}^{r(q+1)\times rn_i} \\ W_i = \begin{pmatrix} W_{i,z} & \cdots & W_{i,r} \end{pmatrix} \in \mathbb{R}^{rn_i\times n_i} \end{matrix} \qquad i = 1, ..., \ell \tag{3.20}$$

where $n_i = \sum_{j=z}^{r} n_{i,j}$ and

$$Q = \begin{pmatrix} Q_1 & \cdots & Q_\ell \end{pmatrix} \in \mathbb{R}^{(q+1)r\times nr} \tag{3.21}$$

$$W = blockdiag \begin{pmatrix} W_1 & \cdots & W_\ell \end{pmatrix} \in \mathbb{R}^{nr \times n} \tag{3.22}$$

Step 4: *Solve the system of equations*

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} QW = 0_{r\times n} \tag{3.23}$$

in terms of the unknown matrices A_i.

Step 5: *Choose the free entries* a_{ij} *of each matrix* A_i *so that* $\det [A(\sigma)] \neq 0$.

Remark 3.3. *In case where in Step 1, there exists no* q *such that* $n = rq$, *the resulting matrix* $A(\sigma)$ *will describe a system of algebraic/difference equations with* $\beta_i^F(k)$ *as part of its solution space, which will include additional vector functions linearly independent from* $\beta_i^F(k)$. *(This holds true for all the algorithms presented in this paper.)*

Remark 3.4. *Every matrix that is* unimodular equivalent *to the polynomial matrix* $A(\sigma)$ *constructed in Algorithm 3.1 gives rise to a model with exactly the same forward behavior with (1.1). That is, all matrices*

$$A_1(\sigma) = U(\sigma)A(\sigma) \tag{3.24}$$

where $U(\sigma)$ *is unimodular satisfy* $A_1(\sigma)\beta_{j,i}(k) = 0$.

4. Modeling the backward behavior of a system described by an AR-Reresentation

In this section we present the backward solution of (1.1) and propose a method of constructing a system that satisfies it.

4.1. Infinite elementary divisors and backward solution space.

Let

$$S^0_{\tilde{\mathrm{A}}(\sigma)}(\sigma) = \tilde{U}_L(\sigma)\tilde{\mathrm{A}}(\sigma)\tilde{U}_R(\sigma) = blockdiag\,[\sigma^{\mu_1}, ..., \sigma^{\mu_r}]$$

be the Smith form of $\tilde{A}(\sigma)$ at zero. Let also $\tilde{U}_R(\sigma) = \begin{bmatrix} \tilde{u}_1(\sigma) & \cdots & \tilde{u}_r(\sigma) \end{bmatrix}$ where $\tilde{u}_j(\sigma) \in R(\sigma)^{r\times 1}$ and $\tilde{u}_j^{(i)}(\sigma)$, $\tilde{A}^{(i)}(\sigma)$ be the i-th derivatives of $\tilde{u}_j(\sigma)$ and $\tilde{A}(\sigma)$ respectively, for $i = 0, 1, ..., \mu_j - 1$ and $j = 2, ..., r$ (since $\mu_1 = 0$). Let

$$x_{j,i} := \tfrac{1}{i!}\tilde{u}_j^{(i)}(0) \quad i = 0, 1, ..., \mu_j - 1 \quad j = 2, ..., r \tag{4.1}$$

and define the vector valued functions

$$\beta^B_{j,\phi}(k) := x_{j,\phi}\delta(N-k) + ... + x_{j,0}\delta(N-(k+\phi)) \tag{4.2}$$

where $j = 2, ..., r$, $\phi = 0, ..., \mu_j - 1$.

Theorem 4.1. [12] *The vector valued functions* $\beta^B_{j,\phi}(k)$ *defined in (4.2) are solutions of (1.1). In addition, let*

$$C_j^B = \begin{bmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,\mu_j-1} \end{bmatrix} \in \mathbb{R}^{r\times\mu_j} \tag{4.3}$$

$$J_j^B := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{\mu_j\times\mu_j} \tag{4.4}$$

and

$$C_B^D := \begin{bmatrix} C_2^D & \cdots & C_r^D \end{bmatrix}, \quad J_B^D := blockdiag \begin{bmatrix} J_2^D & \cdots & J_r^D \end{bmatrix} \tag{4.5}$$

The backward solution space of (1.1) *is spanned by the columns of:*

$$B_B^D = \left\langle C_B^D \left(J_B^D\right)^{N-k} \right\rangle \tag{4.6}$$

and has dimension $\dim B_B^D = \mu$*, where* μ *is defined in* (2.11)*.*

4.2. **Construction of a system with given backward behavior.**

Theorem 4.2. *The vector valued functions* $\beta_{j,\phi}^B(k)$ *defined in* (4.2)*, are solutions of* (1.1) *iff the vectors* $x_{j,i}$ *in* (4.1) *satisfy the system of equations:*

$$\begin{aligned} & A_q x_{j,0} = 0 \\ & A_{q-1} x_{j,0} + A_q x_{j,1} = 0 \\ & \vdots \\ & A_0 x_{j,0} + A_1 x_{x,1} + \ldots + A_q x_{j,q} = 0 \qquad j = k+1, \ldots, r \\ & \vdots \\ & A_0 x_{j,\hat{q}_j - 1} + A_1 x_{j,\hat{q}_j} + \ldots + A_q x_{j,q+\hat{q}_j - 1} = 0 \end{aligned} \tag{4.7a}$$

for the case of infinite zero elementary divisors (i.z.e.d.), i.e. μ_j*,* $j = k+1, \ldots, r$*, or*

$$\begin{aligned} & A_q x_{j,0} = 0 \\ & A_{q-1} x_{j,0} + A_q x_{j,1} = 0 \\ & \vdots \qquad\qquad\qquad\qquad j = 2, \ldots, k \\ & A_{q_j+1} x_{j,0} + \ldots + A_q x_{j,q-q_j-1} = 0 \end{aligned} \tag{4.7b}$$

for the case of infinite pole elementary divisors (i.p.e.d), i.e. μ_j*,* $j = 2, \ldots, k$ *(see* (2.9)*).*

Proof. We show that $\beta_{j,\phi}^B(k)$ are solutions of (1.1) iff (4.7) are satisfied. For $\beta_{j,0}^B(k) = x_{j,0}\delta(N-k)$ we have:

$$A(\sigma) x_{j,0} \delta(N-k) = 0 \Rightarrow$$
$$A_q \delta(N-k-q) x_{j,0} + \ldots + A_1 \delta(N-k-1) x_{j,0} + A_0 \delta(N-k) x_{j,0} = 0$$

and since (1.1) is satisfied for $k \in [0, N-k]$, setting $k = N-q$ we obtain

$$A_q x_{j,0} = 0$$

so the first equation in (4.7) is proven. Now, letting $\beta_{j,1}^B(k) = x_{j,1}\delta(N-k) + x_{j,0}\delta(N-k-1)$, we obtain

$$A(\sigma)\left(x_{j,1}\delta(N-k) + x_{j,0}\delta(N-k-1)\right) = 0 \Rightarrow$$
$$A_q \delta(N-k-q) x_{j,1} + \ldots + A_0 \delta(N-k) x_{j,1} +$$
$$+ A_q \delta(N-k-q-1) x_{j,0} + A_{q-1}\delta(N-k-q) x_{j,0} + \ldots + A_0 \delta(N-k-1) x_{j,0} = 0$$

Again, taking $k = N-q$, we get

$$A_{q-1} x_{j,0} + A_q x_{j,1} = 0$$

so the second equation in (4.7) holds true. Continuing in the same fashion, the rest of the equations in (4.7) can be proven, either for the case of i.z.e.d. or for the case of i.p.e.d. □

Equations (4.7a) and (4.7b) can be rewritten as

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} \underbrace{\begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,q} & \cdots & x_{j,q+\hat{q}_j-1} \\ 0 & x_{j,0} & \cdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & x_{j,0} & \cdots & x_{j,\hat{q}_j-1} \end{pmatrix}}_{Q_j^{B_z}} = 0_{r\times(q+\hat{q}_j)} \quad (4.8a)$$

with $Q_j^{B_z} \in \mathbb{R}^{r(q+1)\times(q+\hat{q}_j)}$, for the case of i.z.e.d. $(j = k+1, ..., r)$ and

$$\begin{pmatrix} A_q & \cdots & A_{q_j+1} \end{pmatrix} \underbrace{\begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,q-q_j-1} \\ 0 & x_{j,0} & \cdots & \vdots \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & x_{j,0} \end{pmatrix}}_{Q_j^{B_p}} = 0_{r\times(q-q_j)} \quad (4.8b)$$

$Q_j^{B_p} \in \mathbb{R}^{r(q+1)\times(q-q_j)}$, for the case of i.p.e.d. $(j = 2, ..., k)$.

Theorem 4.2 is important, because it states that in order for a vector valued function (4.2) to be a solution of $A(\sigma)\beta(k) = 0$, the vectors $x_{j,i}$ need to satisfy (4.8). This system of equations can be used to solve the inverse problem. That is, given a time sequence in the form of $\beta^B_{j,\phi}(k)$, we can solve the system of linear equations (4.8) in terms of the unknown matrices $A_0, A_1, ..., A_q$ and therefore construct the AR-Representation (1.1). These results give rise to the following algorithm for the construction of a system that satisfies a desired backward behavior.

Algorithm 4.1. *Suppose that a finite number of functions of the form*

$$\beta^B_{j,\phi}(k) := x_{j,\phi}\delta(N-k) + ... + x_{j,0}\delta(N-(k+\phi)) \quad (4.9)$$

are given, where $j = 2, ..., r$, $\phi = 0, ..., \mu_j - 1$.

Step 1: *Define* $\mu := \sum_{j=2}^{r} \mu_j$.

If r/μ*, then*

$$q = \frac{\mu}{r}$$

else

$$q = [\frac{\mu}{r}] + 1$$

end if

Step 2: *Construct the matrices* $Q_j^{B_z}$ *and/or* $Q_j^{B_p}$ *defined in* (4.8).

Step 3: *Construct the matrix*

$$Q^B = \begin{pmatrix} Q_2^B & \cdots & Q_r^B \end{pmatrix} \in \mathbb{R}^{r(q+1)\times\mu} \quad (4.10)$$

that can be a combination of the matrices $Q_j^{B_z}$ *and* $Q_j^{B_p}$*, depending on the form of* $\beta^B_{j,i}(k)$ *that are given.*

Step 4: *Solve the system of equations*

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} Q^B = 0_{r\times\mu} \quad (4.11a)$$

or

$$\begin{pmatrix} A_q & \cdots & A_{q_k+1} \end{pmatrix} Q^B = 0_{r\times\mu} \tag{4.11b}$$

in terms of the unknown matrices A_i.

Step 5: *Choose the free entries a_{ij} of each matrix A_i so that* $\det[A(\sigma)] \neq 0$.

From the connection between the Smith form at infinity of $A(\sigma)$ and the Smith form at zero of the dual matrix $\tilde{A}(\sigma)$ in (2.9), it can be seen that the infinite elementary divisors of $A(\sigma)$, that generate the backward solutions of (1.1), are connected to the finite elementary divisors of the dual matrix at $\lambda = 0$, that in turn generate forward solutions for the dual system $\tilde{A}(\sigma)\beta(k) = 0$. More specifically, in [15] the connection between these two behaviors was explicitly given by the following theorem.

Theorem 4.3. [15] *The vector valued functions $\beta^B_{j,\phi}(k)$ defined in (4.2) are solutions of (1.1) iff the vector functions*

$$\tilde{\beta}_{j,\phi}(k) = x_{j,0}\delta(k-\phi) + \cdots + x_{j,\phi}\delta(k) \tag{4.12}$$

where $j = 2, ..., r$, $\phi = 0, ..., \mu_j - 1$, are solutions of the dual system $\tilde{A}(\sigma)\beta(k) = 0$.

Under the above consideration that the backward solutions of (1.1) give rise to forward solutions of its dual system, Remark 3.4 can also be applied here, as follows.

Remark 4.4. *Every polynomial matrix $A_1(\sigma)$ whose dual is unimodular equivalent to the dual of the polynomial matrix $A(\sigma)$ constructed in Algorithm 4.1 gives rise to a model with exactly the same backward behavior with (1.1). That is, all matrices $A_1(\sigma)$ such that:*

$$\tilde{A}_1(\sigma) = U(\sigma)\tilde{A}(\sigma) \tag{4.13}$$

where $U(\sigma)$ is unimodular satisfy $A_1(\sigma)\beta^B_{j,\phi}(k) = 0$.

5. Modeling a System with a Prescribed Forward and Backward Behavior

So far we have provided methods for constructing a system that satisfies a desired forward or backward behavior. Since these methods are functional, we can combine them to give a solution to the general inverse problem. Construct a system of algebraic/difference equations that satisfies a desired forward and a backward behavior. The answer is simple; we can just solve both systems (3.23) and (4.11) and find a solution that satisfies both. As a result, the system produced will have a solution space spanned by the given vector valued functions. These results give rise to the following algorithm.

Algorithm 5.1. *Suppose that a finite number of functions of the form*

$$\beta^F_{j,i}(k) = \lambda_i^k \beta^i_{j,n_{i,j}-1} + \cdots + \binom{k}{n_{i,j}-1}\lambda_i^{k-(n_{i,j}-1)}\beta^i_{j,0} \tag{5.1}$$

$$\beta^F_{j,i}(k) = \delta(k)\beta_{j,n_{i,j}-1} + ... + \delta\left(k-(n_{i,j}-1)\right)\beta_{j,0} \tag{5.2}$$

$$\beta^B_j(k) := x_{j,\mu_j-1}\delta(N-k) + ... + x_{j,0}\delta(N-(k+i)) \tag{5.3}$$

are given.

Step 1: *Define* $n = \sum_{i=1}^{\ell} \sum_{j=z}^{r} n_{ij}$ *and* $\mu := \sum_{j=2}^{r} \mu_j$.
If $r/(n+\mu)$, *then*

$$q = \frac{n+\mu}{r}$$

else

$$q = [\frac{n+\mu}{r}] + 1$$

end if

Step 2: *Construct the matrices* Q *and* W *defined in* (3.21), (3.22), *according to Algorithm* 3.1 *and* Q^B *defined in* (4.10), *according to Algorithm* 4.1.

Step 3: *Solve the system of equations*

$$\begin{array}{c} \begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} QW = 0_{r\times n} \\ AND \\ \begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} Q^B = 0_{r\times \mu} \end{array} \tag{5.4}$$

in terms of the unknown matrices A_i.

Step 4: *Choose the free entries* a_{ij} *of each matrix* A_i *so that* $\det[A(\sigma)] \neq 0$.

Combining the results of Remarks 3.4 and 4.4, we conclude to the following.

Remark 5.1. *Every polynomial matrix* $A_1(\sigma)$ *which is unimodular equivalent to the polynomial matrix* $A(\sigma)$ *constructed in Algorithm* 5.1 *and its dual matrix* $\tilde{A}_1(\sigma)$ *is unimodular equivalent to* $\tilde{A}(\sigma)$, *gives rise to a model with exactly the same forward and backward behavior.*

Example 5.2. *Let the following vector functions*

$$\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\beta_{11}} + \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\beta_{10}} k \qquad \beta_2(k) = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\beta_{21}} 2^k + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\beta_{20}} k2^{k-1}$$

$$\beta_3(k) = \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{x_{33}} \delta(\mathrm{N}-k) + \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{x_{32}} \delta(\mathrm{N}-k-1) +$$

$$+ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x_{31}} \delta(\mathrm{N}-k-2) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x_{30}} \delta(\mathrm{N}-k-3)$$

We want to construct an AR-representation $A(\sigma)\beta(k) = 0$ *that has the prescribed functions in its solution space.*

Step 1: *Since* $\mu = \mu_1 + \mu_2 = 0 + \mu_2 = 4$, $n = n_1 + n_2 = 2 + 2 = 4$ *and* $r = 2$, *from* (2.12) *we have*

$$\begin{array}{c} n + \mu = 4 + 4 = 8 = rq \Rightarrow \\ q = 4 \end{array} \tag{5.5}$$

So

$$A(\sigma) = A_4\sigma^4 + A_3\sigma^3 + A_2\sigma^2 + A_1\sigma + A_0 \in \mathbb{R}^{2\times 2}[\sigma] \tag{5.6}$$

Step 2: *From the coefficients of $\beta_1(k)$ and $\beta_2(k)$, construct the matrices*

$$Q_i = \begin{pmatrix} 4{\lambda_i}^3 I_2 & {\lambda_i}^4 I_2 \\ 3{\lambda_i}^2 I_2 & {\lambda_i}^3 I_2 \\ 2\lambda_i I_2 & {\lambda_i}^2 I_2 \\ I_2 & \lambda_i I_2 \\ 0_2 & I_2 \end{pmatrix} \quad W_i = \begin{pmatrix} \beta_{i0} & 0 \\ \beta_{i1} & \beta_{i0} \end{pmatrix} \quad i = 1, 2 \tag{5.7}$$

and combine them

$$Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} = \left(\begin{array}{cc|cc|cc|cc} 4 & 0 & 1 & 0 & 32 & 0 & 16 & 0 \\ 0 & 4 & 0 & 1 & 0 & 32 & 0 & 16 \\ \hline 3 & 0 & 1 & 0 & 12 & 0 & 8 & 0 \\ 0 & 3 & 0 & 1 & 0 & 12 & 0 & 8 \\ \hline 2 & 0 & 1 & 0 & 4 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 & 0 & 4 & 0 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \tag{5.8}$$

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} = \left(\begin{array}{cc|cc} 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{5.9}$$

From the coefficients of $\beta_3(k)$, since $q = 4$ and $\mu_1 = 0$, we have that μ_2 corresponds either to an infinite pole or an infinite zero elementary divisor. So, either

$$\mu_2 = q - q_2 = 4 \Rightarrow 4 - q_2 = 4 \Rightarrow q_2 = 0 \tag{5.10}$$

or

$$\mu_2 = q + q_2 = 4 \Rightarrow 4 + q_2 = 4 \Rightarrow q_2 = 0 \tag{5.11}$$

which is also accepted. But in order for the matrix dimensions to agree, we will use (4.11b).

$$Q^B = \begin{pmatrix} x_{30} & x_{31} & x_{32} & x_{33} \\ 0 & x_{30} & x_{31} & x_{32} \\ 0 & 0 & x_{30} & x_{31} \\ 0 & 0 & 0 & x_{30} \end{pmatrix} = \left(\begin{array}{c|c|c|c} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{5.12}$$

Step 3 & 4: *Solving the system*

$$\begin{pmatrix} A_4 & A_3 & A_2 & A_1 & A_0 \end{pmatrix} QW = 0_{2\times 4} \tag{5.13}$$

$$\begin{pmatrix} A_4 & A_3 & A_2 & A_1 \end{pmatrix} Q^B = 0_{2\times 4} \tag{5.14}$$

and choosing values for the parameters of the matrices A_i, so that $\det A(s) \neq 0$*, we end up with*

$$A(\sigma) = \begin{pmatrix} \frac{3}{5} - \frac{11\sigma}{5} + 2\sigma^2 - \frac{3\sigma^3}{5} & 2\sigma - \frac{6\sigma^2}{5} - \frac{4\sigma^3}{5} + \frac{3\sigma^4}{5} \\ \frac{1}{10} - \frac{29\sigma}{20} + \frac{7\sigma^2}{4} - \frac{3\sigma^3}{5} & 1 + \sigma - \frac{29\sigma^2}{20} - \frac{11\sigma^3}{20} + \frac{3\sigma^4}{5} \end{pmatrix} \quad (5.15)$$

Note that the Smith forms of $A(\sigma)$ at $\mathbb{C}$ and its dual matrix at zero are

$$S^{\mathbb{C}}_{A(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & (\sigma-1)^2(\sigma-2)^2 \end{pmatrix} \qquad S^{0}_{\tilde{A}(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^4 \end{pmatrix} \quad (5.16)$$

It is easily checked that the vector functions $\beta_i(k)$, $i = 1, 2, 3$ are solutions of the system, i.e. $A(\sigma)\beta_i(k) = 0$. Again, we can find a polynomial matrix $B(\sigma)$ and unimodular matrices $U(\sigma), V(\sigma)$ such that

$$A_1(\sigma) = U(\sigma)A(\sigma) \quad (5.17)$$

$$\tilde{A}_1(\sigma) = V(\sigma)\tilde{A}(\sigma) \quad (5.18)$$

so that $A_1(\sigma)$ satisfies $A(\sigma)\beta_i(k) = 0$. An example is the matrix

$$A_1(\sigma) = \begin{pmatrix} \frac{7}{10} - \frac{73\sigma}{20} + \frac{15\sigma^2}{4} - \frac{6\sigma^3}{5} & 1 + 3\sigma - \frac{53\sigma^2}{20} - \frac{27\sigma^3}{20} + \frac{6\sigma^4}{5} \\ \frac{13}{10} - \frac{117\sigma}{20} + \frac{23\sigma^2}{4} - \frac{9\sigma^3}{5} & 1 + 5\sigma - \frac{77\sigma^2}{20} - \frac{43\sigma^3}{20} + \frac{9\sigma^4}{5} \end{pmatrix} \quad (5.19)$$

with

$$U(\sigma) = U = V(\sigma) = V = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad (5.20)$$

6. Notes on the Power of a model

The notion of power in modeling has been introduced by [22] and later studied in [24, 25, 26]. The power of a model is defined as the ability of the constructed model (i.e. the AR-representation) to describe the given behavior, i.e. the given data, but as little else as possible. So if we define as B the behavior of the system we have constructed, i.e. the complete set of vector valued functions that satisfy it:

$$\mathrm{B} = \{w : \mathbb{N} \to \mathbb{R}^n | \mathrm{A}(\sigma)w(k) = 0\} \quad (6.1)$$

or equivalently

$$\mathrm{B} = \ker \mathrm{A}(\sigma) \quad (6.2)$$

we do not simply desire this behavior to include the given functions. This should obviously be the aim of the modelling procedure, but the optimal goal for the contructed model is to have no other behavior, linearly independent from the prescribed. So for any other model with behavior B_1 we want

$$\{B \text{ more powerful than } B_1\} \Leftrightarrow \{B \subseteq B_1\} \quad (6.3)$$

Now, as mentioned previously, for a given number of vector valued functions, the system created by the proposed algorithms may still include extra forward/backward behavior if equation $n + \mu = rq$ is not satisfied. In this case the system model is not the most powerful model (and no such model can be created for a square and regular system matrix). These facts give rise to the following theorem.

Remark 6.1. *Given the following vector valued functions*

$$\beta^F_{j,i}(k) = \lambda_i^k \beta^i_{j,n_{i,j}-1} + \cdots + \binom{k}{n_{i,j}-1} \lambda_i^{k-(n_{i,j}-1)} \beta^i_{j,0} \quad (6.4)$$

$$\beta^F_{j,i}(k) = \delta(k)\beta_{j,n_{i,j}-1} + ... + \delta\left(k - (n_{i,j}-1)\right)\beta_{j,0} \quad (6.5)$$

$$\beta_j^B(k) := x_{j,\mu_j-1}\delta(N-k) + ... + x_{j,0}\delta(N-(k+i)) \tag{6.6}$$

let $n = \sum_{i=1}^{\ell}\sum_{j=z}^{r} n_{ij}$ *and* $\mu := \sum_{j=2}^{r} \mu_j$.

The system $A(\sigma)\beta(k) = 0$ *constructed by the proposed Algorithm 5.1, corresponding to the behavior* $\mathrm{B} = \ker \mathrm{A}(\sigma)$ *is the most powerful model that describes the above vector valued functions iff* $\exists\, q \in \mathbb{N}$ *such that*

$$n + \mu = rq \tag{6.7}$$

If this is not the case, then choosing $q = [\frac{n+\mu}{r}] + 1$ *in Algorithm 5.1, results in* $n + \mu < rq$ *and so the behavior of the system includes to* $rq - n - \mu$ *additional solutions, since the dimension of the solution space is always equal to* rq.

An example where the constructed system is not the most powerful model is given below.

Example 6.2. *Let the following vector valued functions*

$$\beta_1(k) = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\beta_{12}} 2^k + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\beta_{11}} k2^{k-1} + \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\beta_{10}} \frac{k(k-1)}{2} 2^{k-2}$$

We want to construct an AR-representation $A(\sigma)\beta(k) = 0$ *that has the prescribed functions in its solution space.*

Step 1: *Since* $n = n_1 = 3$, $\mu = 0$ *and* $r = 2$, *from (2.12) we have* $n + 0 = 2q \Rightarrow q = 3/2$. *So set* $q = [\frac{3}{2}] + 1 = 2$ *and we have*

$$A(\sigma) = A_2\sigma^2 + A_1\sigma + A_0 \in \mathbb{R}^{2\times 2}[\sigma] \tag{6.8}$$

Step 2: *Construct the matrices* Q_1 *and* W_1*:*

$$Q_1 = \begin{pmatrix} I_2 & 2\cdot 2I_2 & 2^2 I_2 \\ 0 & I_2 & 2I_2 \\ 0 & 0 & I_2 \end{pmatrix} = \left(\begin{array}{cc|cc|cc} 1 & 0 & 4 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 & 0 & 4 \\ \hline 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{6.9}$$

$$W_1 = \begin{pmatrix} \beta_{10} & 0 & 0 \\ \beta_{11} & \beta_{10} & 0 \\ \beta_{12} & \beta_{11} & \beta_{10} \end{pmatrix} = \left(\begin{array}{c|c|c} 2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 4 & 2 & 0 \\ 1 & 0 & 0 \\ \hline 2 & 4 & 2 \\ 3 & 1 & 0 \end{array}\right) \tag{6.10}$$

Step 4: *Solve the system*

$$\begin{pmatrix} A_2 & A_1 & A_0 \end{pmatrix} Q_1 W_1 = 0_{2\times 3} \tag{6.11}$$

where

$$A_i = \begin{pmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{pmatrix} \tag{6.12}$$

Step 5: *The matrices that we end up with are*

$$A_0 = \begin{pmatrix} -a_{02} - 4(a_{12} + 3a_{22}) & a_{02} \\ -a_{04} - 4(a_{14} + 3a_{24}) & a_{04} \end{pmatrix} \tag{6.13}$$

$$A_1 = \begin{pmatrix} \frac{3}{2}a_{02} + 5a_{12} + 14a_{22} & a_{12} \\ \frac{3}{2}a_{04} + 5a_{14} + 14a_{24} & a_{14} \end{pmatrix} \tag{6.14}$$

$$A_2 = \begin{pmatrix} \frac{1}{2}(-a_{02} - 3a_{12} - 8a_{22}) & a_{22} \\ \frac{1}{2}(-a_{04} - 3a_{14} - 8a_{24}) & a_{24} \end{pmatrix} \tag{6.15}$$

and the matrix $A(\sigma)$ *has a determinant*

$$\det A(\sigma) =$$
$$(\sigma - 2)^3(a_{04}a_{12} - a_{02}a_{14} + 3(a_{04}a_{22} - a_{02}a_{24}) + (a_{04}a_{22} + 3a_{14}a_{22} - a_{02}a_{24} - 3a_{12}a_{24})\sigma)$$

which is a polynomial of degree equal to $4 = rq = 2 \cdot 2$. *So it is obvious that the matrix has an extra zero and thus an extra solution, as it was expected, since* $rq - n - \mu = 4 - 3 = 1$.

For example, by choosing $a_{22} = a_{12} = a_{02} = 1$, $a_{14} = a_{24} = 0$, $a_{04} = 2$ *we get the matrix*

$$A(\sigma) = \begin{pmatrix} -17 + \frac{41\sigma}{2} - 6\sigma^2 & 1 + \sigma + \sigma^2 \\ -2 + 3\sigma - \sigma^2 & 2 \end{pmatrix} \tag{6.16}$$

with Smith form

$$S^{\mathbb{C}}_{A(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & (\sigma + 4)(\sigma - 2)^3 \end{pmatrix} \tag{6.17}$$

So $A(\sigma)$ *has an additional finite elementary divisor, i.e.* $(\sigma+4)$. *So for this matrix the new number of f.e.d and i.e.d. is* $n' = 4$ *and* $\mu = 0$. *Following the procedure described in Subsection 3.1, we find that this divisor gives rise to the solution*

$$\beta_2(k) = \begin{pmatrix} 1 \\ 15 \end{pmatrix} (-4)^k \tag{6.18}$$

which is linearly independent from $\beta_1(k)$.

On the other hand, one may assume that by choosing the appropriate values of the free parameters in order to eliminate the coefficient of σ *in the extra polynomial of the determinant, while still keeping* $\det A(\sigma) \neq 0$, *will give a simple solution to the problem of undesired behavior. This is not the case, since this will lead to undesired backward behavior. For example, by choosing* $a_{12} = 1$, $a_{02} = a_{14} = a_{22} = a_{24} = 0$, $a_{04} = 2$ *we get*

$$A(\sigma) = \begin{pmatrix} -4 + 5\sigma - \frac{3\sigma^2}{2} & \sigma \\ -2 + 3\sigma - \sigma^2 & 2 \end{pmatrix} \tag{6.19}$$

with

$$S^{\mathbb{C}}_{A(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & (\sigma - 2)^3 \end{pmatrix} \qquad S^{0}_{\tilde{A}(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \tag{6.20}$$

The Smith form of the dual matrix at $\sigma = 0$ *implies that in this case the matrix* $A(\sigma)$ *has an additional infinite elementary divisor, so here we have* $n' = 3$ *and* $\mu' = 1$. *The existence of an infinite elementary divisor implies the existence of additional backward behavior for the above system. Thus, we see that no matter what the values of the free variables* a_{ij} *will be, the system will exhibit additional behavior.*

As an alternative, one may proceed to construct a non-regular *system that satisfies the prescribed behavior, i.e. a system with* $A(\sigma) \in \mathbb{R}^{r\times m}$ *and* $n \neq m$ *or with* $A(\sigma) \in \mathbb{R}^{r\times r}$ *and* $\det A(\sigma) = 0$.

Step 1: *Under the assumption that the constructed system can be non-square, taking r=1 and n=3, we find* $n = rq \Rightarrow q = 3$ *(see also [16]). So*

$$A(\sigma) = A_3\sigma^3 + A_2\sigma^2 + A_1\sigma + A_0 \in \mathbb{R}^{1\times 2}[\sigma] \tag{6.21}$$

Step 2: *For the above system, the matrix* W_1 *remains the same, while* Q_1 *is*

$$Q_1 = \begin{pmatrix} 3\cdot 2I_2 & 3\cdot 2^2 I_2 & 2^3 I_2 \\ I_2 & 2\cdot 2I_2 & 2^2 I_2 \\ 0 & I_2 & 2I_2 \\ 0 & 0 & I_2 \end{pmatrix} = \left(\begin{array}{cc|cc|cc} 6 & 0 & 12 & 0 & 8 & 0 \\ 0 & 6 & 0 & 12 & 0 & 8 \\ \hline 1 & 0 & 4 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 & 0 & 4 \\ \hline 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{6.22}$$

Step 4: *Solve the system*

$$\begin{pmatrix} A_3 & A_2 & A_1 & A_0 \end{pmatrix} Q_1 W_1 = 0_{1\times 3} \tag{6.23}$$

where

$$A_i = \begin{pmatrix} a_{i1} & a_{i2} \end{pmatrix} \tag{6.24}$$

Step 5: *The matrices that we end up with are*

$$A_0 = \begin{pmatrix} -a_{02} - 4a_{12} - 12a_{22} - 8a_{31} - 32a_{32} & a_{02} \end{pmatrix} \tag{6.25}$$

$$A_1 = \begin{pmatrix} \frac{3a_{02}}{2} + 5a_{12} + 14a_{22} + 12a_{31} + 36a_{32} & a_{12} \end{pmatrix} \tag{6.26}$$

$$A_2 = \begin{pmatrix} -\frac{a_{02}}{2} - \frac{3a_{12}}{2} - 4a_{22} - 6a_{31} - 10a_{32} & a_{22} \end{pmatrix} \tag{6.27}$$

$$A_3 = \begin{pmatrix} a_{31} & a_{32} \end{pmatrix} \tag{6.28}$$

and the constructed matrix $A(\sigma)$ *satisfies* $A(\sigma)\beta(k) = 0$. *What must be noted though is that since this procedure has led to the construction of a non-regular system, the solutions* $\beta_i(k)$ *of the system could be attributed to either its f.e.d. structure or its right null space. Indeed, the matrix* $A(\sigma)$ *has a Smith form*

$$S^{\mathbb{C}}_{A(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{6.29}$$

That it, as [12] demonstrates, non-regular systems exhibit an infinite number of forward and backward solutions due to the right null space of $A(\sigma)$. *So in this case, the constructed system will include additional behavior that is undesired.*

7. Conclusions

A novel method has been proposed for constructing an AR-Representation that satisfies a prescribed forward and backward behavior, given in the form of vector valued functions. It was shown (see Example 6.2) that this method can also be used to construct non-regular systems. Thus, the proposed method is more versatile than previous ones for continuous or discrete time systems (see [8, 13, 15]) that only worked for square matrices. The results presented in this work can also be used with minor adjustments to the case of continuous time systems, where smooth and impulsive behaviors are of interest.

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A DYNAMIC MODEL FOR HIV INFECTION

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ABSTRACT. A dynamical model that describes the effect of the HIV virus on the immune system is presented. The effect of introducing antiretroviral therapy on the model, consisting of RTIs and PIs, is investigated, along with the result of undesired treatment interruption. The effect of both drugs can be combined into a single input that further simplifies the model. Furthermore, the system is linearized around the equilibrium, leading to a system of linear differential equations of first order that can be integrated into courses of control systems engineering in higher education.

1. INTRODUCTION

According to the most recent HIV/AIDS surveillance report in Greece, on October of 2015 [7], the Hellenic Center For Disease Control & Prevention (H.C.D.C.P.) has so far reported 15.109 positive HIV infections. Of these, 3.782 have already developed AIDS and around 7.700 are subject to antiretroviral therapy (ART). The number of deaths resulting from the infection amounts to 2.562. According to the H.C.D.C.P. 2014 report [8], the largest portion of HIV cases has been diagnosed in men who had sex with men (46.2%), followed by the categories of heterosexual sexual contact (21.3%) and injecting drug users (10.8%).

More specifically, during the period of 2011-2013, there was a big rise in the number of cases in injecting drug users, that was followed by a steady decrease during the last two years. Yet, as the Office for HIV and Sexually Transmitted Diseases emphasizes, although the last data on the decrease infections are positive, they should not be considered comforting. There must be constant actions for the awareness of both the high risk groups and the general population.

Motivated by similar and even more alarming statistics in South Africa, the University of Pretoria, being aware of the fact that the student population generally falls into the high risk groups, mainly due to lack of awareness, decided to organise an action to inform the students about the problem. The department of Electrical, Electronic and Computer Engineering, the department of Telematic Learning and Education Innovation and the Center for the Study of AIDS came together and developed a CD [4, 5], with the aim of presenting a model for the HIV infection from a control theory perspective. Their aim was to present the problem through

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a mathematical model that would introduce the students to the field of control systems engineering, motivating them at the same time to learn more about this sensitive subject.

Base on this innovative idea by the University of Pretoria, we propose an analytical description of the dynamic model for HIV infection, with the purpose of fulfilling two different objectives. First, to present a detailed control engineering problem that can be implemented in a vast variety of undergraduate courses in the field of dynamical systems, thus making the syllabus much more interesting through the perspective of real life applications. This study, as will be shown later, is subject to extensive research [1–3, 6, 9, 10, 12–15, 18–22, 24, 25] and can be extended to master and doctoral studies. Secondly, the awareness abong students on the subject should be a natural consequence of taking such a subject.

2. The dynamical model of HIV

The Human Immunodeficiency Virus (HIV) acts by attacking the immune system, causing its progressive failure over time and its collapse after years (when no treatment is administered). The virus can be transmitted mainly through sexual intercourse without protection. In addition the virus can be spread through sharing needles in drug users and in health care accidents, through blood, organ or sperm donations and from mother to child during pregnancy or birth [6].

The virus acts by infecting the CD4+ T cells. In the initial days of the infection, the virus rapidly multiplies and as a result in the first 2-12 weeks the patient develops general flu-like symptoms like fever, chills, rashes, night sweats, sore throat, fatigue and swollen lymph nodes. This is called the *acute HIV infection stage*. The spread of the virus activates the immune system to fight of the infection. This leads after a period of 12-15 weeks to the suppression of the virus spread and the stabilization of the immune system.

Now, the patient enters the *clinical latency stage*, also called the *chronic HIV infection*. During this stage there is a balance between healthy CD4+ cells and viral load, so the virus is still active but is repressed by the immune system and reproduces at very low levels. This stage may last as long as 10 years for patients who do not take medication and up to many decades for patients who are properly administered to antiretroviral therapy. Eventually, through the chronic deterioration, the immune system becomes weak and vulnerable, making the individual vulnerable to opportunistic infections. This is the final stage of the HIV infections and is called the Acquired ImmunoDeficiency Syndrome (AIDS). It should be noted though that not all HIV positive people advance to this stage.

A simple model that describes the effect of the HIV to the immune system can be constructed by describing the interactions between healthy CD4+ T cells, infected CD4+ cells and the viral load, see Figure 1. Healthy CD4+ cells are produced by the thymus at a constant rate of s and die at a rate d. They are infected by the virus at a rate that is proportional to the product of their number and the viral load. The effectiveness of the infection is given by a constant β. The infected CD4+ cells result from the infection of healthy cells and die at a constant rate m_2. Free virus particles, known as virions are produced from infected CD4+ cells at a rate k and die at a rate m_1 [2, 4, 5, 14, 15].

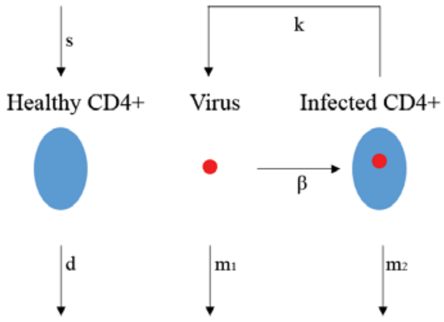

FIGURE 1. Interaction of HIV and CD4+ cells.

These interactions between healthy CD4+ cells, infected CD4+ and free virions can be described by the following system of nonlinear equations

$$\dot{T}(t) = s - dT(t) - \beta T(t)v(t) \tag{2.1a}$$
$$\dot{T}^*(t) = \beta T(t)v(t) - m_2 T^*(t) \tag{2.1b}$$
$$\dot{v}(t) = kT^*(t) - m_1 v(t) \tag{2.1c}$$

where $T(t)$ the number of healthy CD4+, $T^*(t)$ the number of infected CD4+ and $v(t)$ the number of virions, also known as the viral load. Typical values for the parameters of the system are given in Table 1, according to [2,4,5].

t	Time	Days
d	Death rate of uninfected T cells	0.02 per day
k	Rate of virions produced per infected T cell	100 counts/cell
s	Production rate of uninfected T cells	100mm^{-3}/day
β	Infectivity rate of virions	$2.4\times10^{-5}\text{mm}^{-3}$/day
m_1	Death rate of virus	2.4/day
m_2	Death rate of infected T cells	0.24/day

TABLE 1. Typical values for the system parameters.

A typical progression for the disease is shown in Figure 2. It is clear that after initial infection, there is a rise in the infected CD4+ cells and after the reaction by the immune system,the system is stabilized and we have a pass to the clinical latency stage.

It should also be noted that the system always ends up in the same equilibrium point, regardless of initial condition of the patient. This can be seen in Figure 3, which shows multiple trajectories for varying initial conditions (note that some may correspond to unrealistic data).

3. Antiretroviral treatment

Highly active antiretroviral therapy, or HAART, consists of taking multiple drugs with different antiviral targets that maintain the function of the immune system and suppress the virus. Two basic categories of antiretroviral drugs are the reverse transcriptase inhibitors or RTIs and protease inhibitors or PIs. There are also other

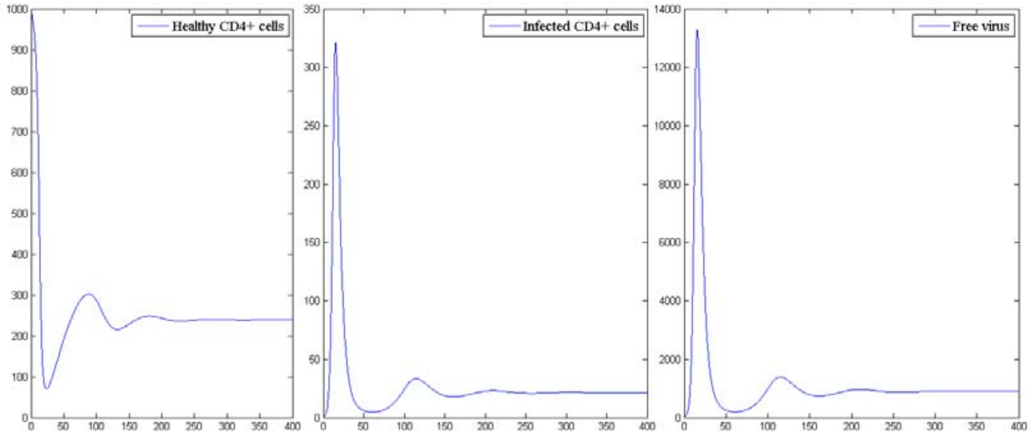

FIGURE 2. Disease progression.

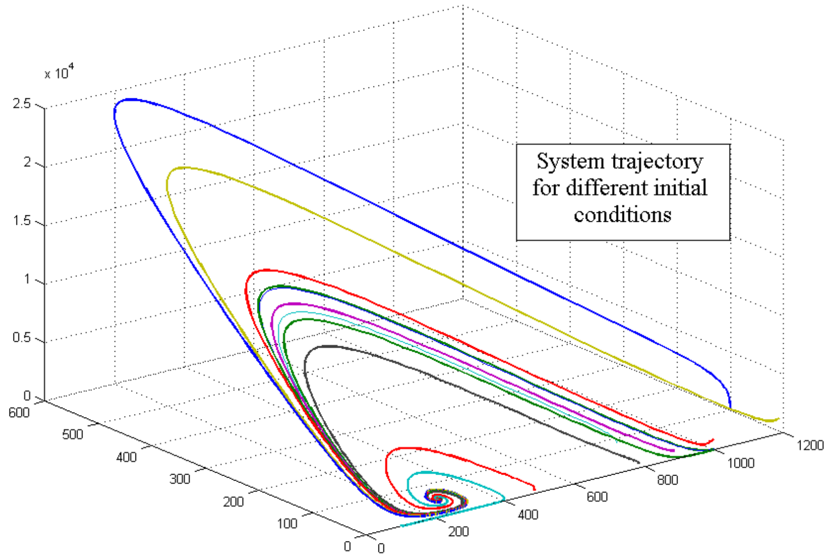

FIGURE 3. System trajectories for different initial conditions.

drug categories like non-nucleoside reverse-transcriptase inhibitors (NNRTIs), nucleoside reverse transcriptase inhibitors (NRTIs) and fusion/entry inhibitors. Since though most of the current bibliography focuses on RTIs and PIs, we adopt this line of analysis. RTIs act by blocking the infection of new T cells while PIs prevent the production of new virions. Taking into consideration the effect of these artiretroviral drugs, model (2.1) takes the form [2,4,5,14,15]:

$$\dot{T}(t) = s - dT(t) - (1 - u_1(t))\,\beta T(t)v(t) \tag{3.1a}$$

$$\dot{T}^*(t) = (1 - u_1(t))\,\beta T(t)v(t) - m_2 T^*(t) \tag{3.1b}$$

$$\dot{v}(t) = (1 - u_2(t))\,kT^*(t) - m_1 v(t) \tag{3.1c}$$

where the terms $(1-u_1(t))$ and $(1-u_2(t))$ represent the effectiveness of RTIs and PIs respectively (for $u_{1,2}=0$ the drug is not administered, while for $u_{1,2}=1$ the treatment is 100% effective, which of course is not achievable). Here, by trying different combinations of intensity for the two drugs we can observe their effect on the viral load, as shown in Figure 4. Indeed, it can be seen that the viral load is successfully suppressed. The treatment is initiated at the 150th day.

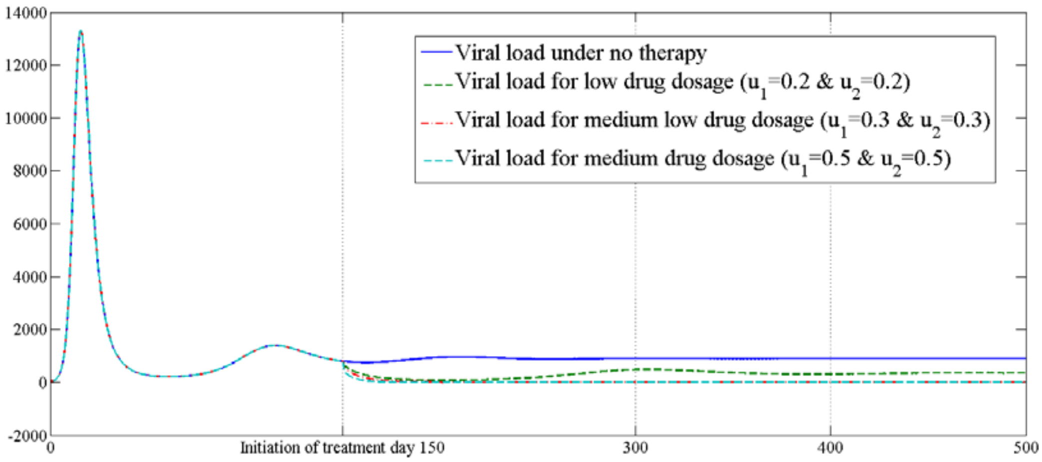

FIGURE 4. Viral load for different drug dosages.

What should also be noted is that the drugs should be taken continuously and with no interruptions, so that the virus is always suppressed and is not given the chance to mutate. In case the treatment is interrupted, there is a big possibility that the virus will regress back to high levels. This is something that can be confirmed from the model (3.1). Indeed, as can be seen in Figure 5, if the treatment starts at day 150 but is terminated at day 400, the virus load may stay low for a maximum of 150 more days, depending on the effectiveness of the drugs, but rises back up afterwards. The issue of treatment interruption has been clinically studied in [24].

As a further simplification of the model (3.1), the two inputs can be combined into a single one, that acts on the third differential equation. More specifically in [2,14,15], it was shown after clinical studies that the effects of RTI and PI drugs cannot be considered decoupled. Furthermore, the combined treatment seems to be much more effective on the parameter k than in β. Taking into account these observations, the nonlinear model (3.1) takes the form

$$\dot{T}(t) = s - dT(t) - \beta T(t)v(t) \tag{3.2a}$$

$$\dot{T}^*(t) = \beta T(t)v(t) - m_2 T^*(t) \tag{3.2b}$$

$$\dot{v}(t) = (1-u(t))\, kT^*(t) - m_1 v(t) \tag{3.2c}$$

where the parameter $u(t)$ denotes the effectiveness of the combined treatment. The systems response for a single input model is shown in Figure 6. A notable difference that we observe though in constrast to the two input model (3.1) is that the viral load exhibits different and larger overshoot for different drug dosages, after the interruption of treatment.

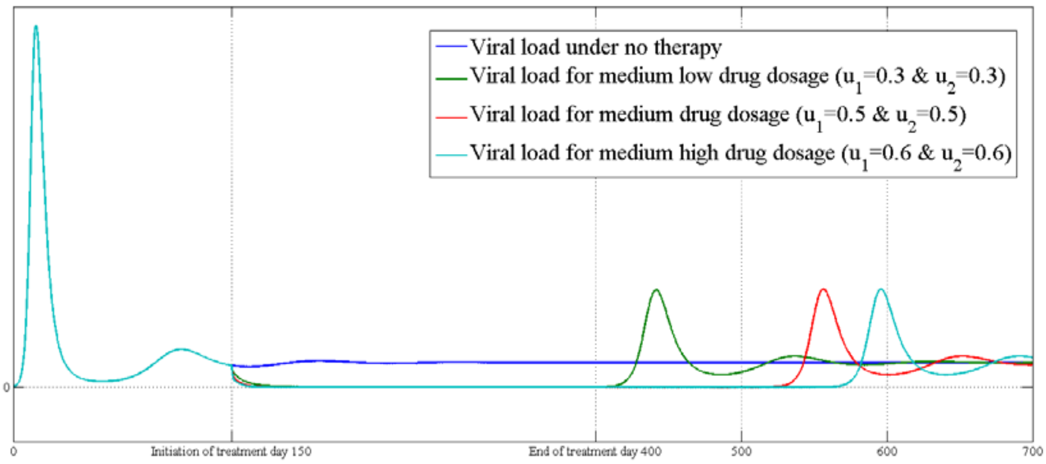

FIGURE 5. Viral load after treatment interruption.

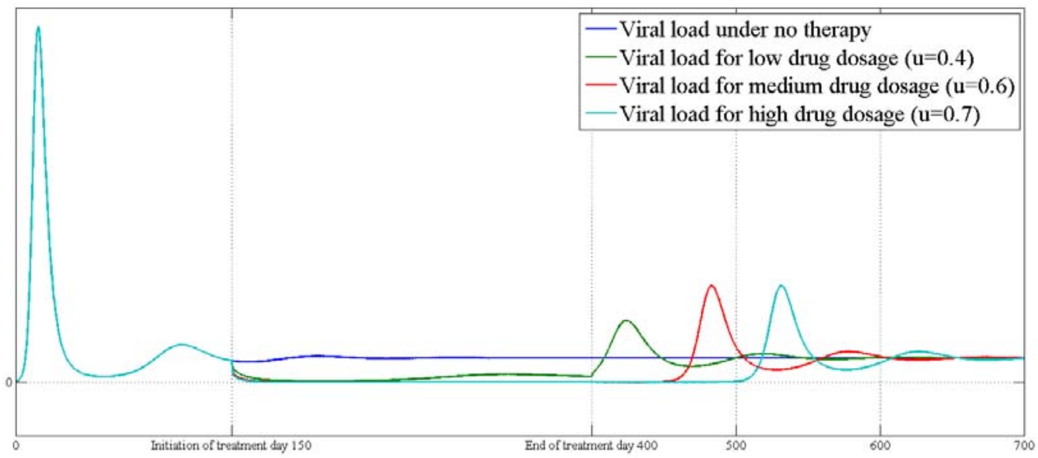

FIGURE 6. Viral load for the single input system.

4. LINEARIZATION

The analysis of the nonlinear systems (2.1),(3.1),(3.2) is a very complex and challenging task. For this reason, our aim is to determine the dynamical behavior of the system around its equilibrium points.

Definition 4.1. *[23] A point $x^* \in \mathbb{R}^n$ is an equilibrium point for the system $\dot{x} = f(t, x)$ if $f(x^*) = 0$ for all $t \geq 0$.*

So, in order to specify the equilibrium points of (2.1), we solve the system of equations

$$0 = s - dT(t) - \beta T(t)v(t) \tag{4.1a}$$
$$0 = \beta T(t)v(t) - m_2 T^*(t) \tag{4.1b}$$
$$0 = kT^*(t) - m_1 v(t) \tag{4.1c}$$

In case where $T^* = v = 0$, the equilibrium is

$$\begin{pmatrix} \frac{s}{d} & 0 & 0 \end{pmatrix} \tag{4.2}$$

and in case where $v \neq 0$ the equilibrium is

$$\begin{pmatrix} \frac{m_1 m_2}{k\beta} & \frac{s}{m_2} - \frac{d m_1}{\beta k} & \frac{ks}{m_1 m_2} - \frac{d}{\beta} \end{pmatrix} = \begin{pmatrix} 240 & 21.6667 & 902.778 \end{pmatrix} \tag{4.3}$$

The first equilibrium corresponds to a healthy uninfected individual, so it is not of interest. The second equilibrium corresponds to the equilibrium point after the patients enters the clinical latency stage. To linearise the system around the equilibrium, we first compute the Jacobian of the system, which is given by

$$J(f) = \begin{pmatrix} -d - \beta v & 0 & -\beta T \\ \beta v & -m_2 & \beta T \\ 0 & k & -m_1 \end{pmatrix} \tag{4.4}$$

Computing the eigenvalues of the Jacobian, we find that they are all stable, and thus the equilibrium point is hyperbolic [11]. This, in combination with the Hartman-Grobman theorem [23] guarantees that the linearization is possible and that the linearized system preserves the qualitative properties of the nonlinear system around the equilibrium. With the addition of inputs, we define

$$\tilde{f}_1(T, T^*, v, u_1, u_2) = s - dT(t) - (1 - u_1)\beta T(t)v(t) \tag{4.5a}$$

$$\tilde{f}_2(T, T^*, v, u_1, u_2) = (1 - u_1)\beta T(t)v(t) - m_2 T^*(t) \tag{4.5b}$$

$$\tilde{f}_3(T, T^*, v, u_1, u_2) = (1 - u_2)kT^*(t) - m_1 v(t) \tag{4.5c}$$

The linearized system is [11]:

$$\begin{pmatrix} \dot{T} \\ \dot{T}^* \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial T} & \frac{\partial \tilde{f}_1}{\partial T^*} & \frac{\partial \tilde{f}_1}{\partial v} \\ \frac{\partial \tilde{f}_2}{\partial T} & \frac{\partial \tilde{f}_2}{\partial T^*} & \frac{\partial \tilde{f}_2}{\partial v} \\ \frac{\partial \tilde{f}_3}{\partial T} & \frac{\partial \tilde{f}_3}{\partial T^*} & \frac{\partial \tilde{f}_3}{\partial v} \end{pmatrix} \begin{pmatrix} T \\ T^* \\ v \end{pmatrix} + \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial u_1} & \frac{\partial \tilde{f}_1}{\partial u_2} \\ \frac{\partial \tilde{f}_2}{\partial u_1} & \frac{\partial \tilde{f}_2}{\partial u_2} \\ \frac{\partial \tilde{f}_3}{\partial u_1} & \frac{\partial \tilde{f}_3}{\partial u_2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.6}$$

computing the values of the above matrices for

$$\begin{pmatrix} T & T^* & v & u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 240 & 21.6667 & 902.778 & 0 & 0 \end{pmatrix} \tag{4.7}$$

we end up with the state space system

$$\begin{pmatrix} \dot{T} \\ \dot{T}^* \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} -0.0417 & 0 & -0.0058 \\ 0.0217 & -0.24 & 0.0058 \\ 0 & 100 & -2.4 \end{pmatrix}}_{A} \begin{pmatrix} T \\ T^* \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 5.2 & 0 \\ -5.2 & 0 \\ 0 & -2166.67 \end{pmatrix}}_{B} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.8a}$$

$$y = \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{C} \begin{pmatrix} T \\ T^* \\ v \end{pmatrix} \tag{4.8b}$$

where we chose as an output the viral load. The next step in the analysis of the state space system (4.8) is to compute its transfer function, that gives the connection between the output and each input [16]. It is computed as

$$G(s) = C(sI_3 - A)^{-1}B = \tag{4.9}$$

$$= \begin{pmatrix} \frac{-520\text{s} - 10.4}{s^3 + 2.682s^2 + 0.1061s + 0.01242} & \frac{-2167s^2 - 610.4s - 21.68}{s^3 + 2.682s^2 + 0.1061s + 0.01242} \end{pmatrix} \tag{4.10}$$

The controllability matrix is given by

$$\mathcal{L} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \tag{4.11}$$

$$= \begin{pmatrix} 5.2 & 0 & -0.216667 & 12.48 & 3.00423 & -30.472 \\ -5.2 & 0 & 1.36067 & -12.48 & -3.32645 & 33.2176 \\ 0 & -2166.67 & -520. & 5200. & 1384.07 & -13728. \end{pmatrix} \tag{4.12}$$

and it has full rank and so the system is controllable. Thus, the system can be compensated through the use of open or closed loop controllers.

If we consider the single input nonlinear system (3.2), as presented in the previous section, then following the same procedure we end up in the state space system (4.8), where the new matrix $\bar{B}$ consists of just the second column of B.

Although the linear model (4.8) is a simplification of (2.1) and only captures its dynamical qualities around the equilibrium, can be used as a basis for the demonstration of a plethora of problems of control systems engineering. Such topics include the state feedback of the system (4.8) and the computation of its gain margin, the design of PID controllers for the reduction of the viral load, the evaluation of the sampling time through Bode diagrams to decide the frequency in which the patient should be tested and many more.

5. Conclusions

We presented a fundamental nonlinear model that describes the HIV infection. The effect of antiretroviral treatment was studied under variable drug effectiveness. Then a linera state space model was developed to further simplify the dynamic behavior of the system and various control engineering problems where proposed like the design of controllers for its compensation. Every part of this work can be potentially integrated into the syllabus of linear and nonlinear dynamical systems courses and can be combined with the use of computer software like Matlab [16, 17] to simulate the above models. Further research on the field of HIV/AIDS infection constitutes the study of more complex nonlinear models [1, 3, 9, 12, 19, 25] that describe more accurately the complicated nature of the virus, the problem of feedback linearization of the nonlinear system [2, 14, 15, 20] and the use of time variant or even impulsive inputs for its control [2, 14, 15, 21].

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TOPOLOGICAL OBSTRUCTIONS OF ISOMETRIC IMMERSIONS

CHRISTOS ONTI

ABSTRACT. According to Nashs embedding theorem, every Riemannian manifold admits an isometric immersion into a Euclidean space with suffciently high codimension. On the other hand, when the codimension is low topological restrictions do appear. For example, every closed (compact, without boundary) positively curved hypersurface in Euclidean space is diffeomorphic to a sphere. The aim of our talk is to present this type of topological obstructions of isometric immersions of low codimension.

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TOPOLOGICAL OBSTRUCTIONS OF ISOMETRIC IMMERSIONS

KLEANTHIS POLYMERAKIS

Abstract. The problem of rigidity or deformability of isometric immersions and the study of the moduli space of a deformable immersion, are of the most central in the Theory of Isometric Immersions. In this talk, basic results will be presented that provide partial answers to the rigidity problem, such as Beez-Killing's and Do Carmo-Dajczer's theorems, as well as results concerning deformable immersions.

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RECOLLEMENTS OF DERIVED MODULE CATEGORIES

CHRYSOSTOMOS PSAROUDAKIS

Abstract. Recollements of abelian/triangulated categories are exact sequences of abelian/triangulated categories where both the inclusion and the quotient functors have left and right adjoints. They appear quite naturally in various settings and are omnipresent in representation theory. Recollements in which all categories involved are module categories (abelian case) or derived categories of module categories (triangulated case) are of particular interest. In the abelian case, the standard example is the recollement induced by the module category of a ring R with an idempotent element e, and in the triangulated case the standard example is given as the derived counterpart of this recollement of module categories when the ideal ReR is stratifying. The latter recollement is called stratifying. The aim of this talk is two-fold. First, we classify, up to equivalence, recollements of abelian categories whose terms are equivalent to module categories. Then, we provide necessary and sufficient conditions for a recollement of derived categories of module categories to be equivalent to a stratifying one. In particular, we show that every derived recollement of a finite dimensional hereditary algebra is equivalent to a stratifying one.

This is joint work with Jorge Vitória (arXiv:1511.02677).

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THE ASSOCIATED LIE RING OF A FORMANEK-PROCESI GROUP

H. SEVASLIDOU

ABSTRACT. Let G be a group. For a positive integer c, we write $\gamma_c(G)$ for the c-th term of the lower central series of G. Let

$$\mathrm{gr}(G) = \bigoplus_{c\geq 1} \gamma_c(G)/\gamma_{c+1}(G)$$

be the associated Lie ring of G. Motivated by a recent work of Metaftsis and Papistas regarding the McCool group M_3, we describe the associated Lie ring of the Formanek-Procesi group

$$H = \langle t, a, b : (t, a, a) = (t, b, b) = (t, b, a) = (t, a, b) = 1\rangle.$$

It is important to mention that the above group is not a linear group. It is constructed by Formanek and Procesi in order to show that the automorphism group of a free group of finite rank > 3 is not linear. Furthermore, Bardakov and Mikhailov proved that that the group of IA-automorphisms of F_3 is not linear, since it contains such a group as a subgroup.

1. Introduction

By "Lie algebra", we mean Lie algebra over the ring of integers $\mathbb{Z}$. Let G be a group. We denote by (a, b) the commutator $(a, b) = a^{-1}b^{-1}ab$. For a positive integer c, let $\gamma_c(G)$ be the cth term of the lower central series of G. The (restricted) direct sum of the quotients $\gamma_c(G)/\gamma_{c+1}(G)$ is the *associated Lie algebra* of G,

$$Ł(G) = \bigoplus_{c\geq 1} \gamma_c(G)/\gamma_{c+1}(G).$$

The Lie bracket multiplication in $Ł(G)$ is defined as

$$[a\gamma_{c+1}(G), b\gamma_{d+1}(G)] = (a, b)\gamma_{c+d+1}(G),$$

with $a \in \gamma_c(G)$, $b \in \gamma_d(G)$ and $(a, b) \in \gamma_{c+d}(G)$ and extends the multiplication linearly. The Formanek-Procesi groups are HNN extensions of the form

$$\mathcal{H}(G) = \langle G \times G, t : t(g, g)t^{-1} = (1, g) \text{ for all } g \in G\rangle.$$

Let F_n be free group of finite rank n, with $n \geq 2$. We point that

$$\bigcap_{c\geq 1} \gamma_c(F_n) = \{1\},$$

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that is, F_n is residually nilpotent. It is known that the Lie algebra $\L(F_n)$ is free of rank n.

We are interested in the associated Lie algebra $\L(\mathcal{H}(F_n))$. The group $\mathcal{H}(F_n)$ has the following presentation

$$\mathcal{H}(F_n) = \langle t, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n : b_j a_i b_j^{-1} = a_i, b_i a_{n+1} b_i^{-1} = a_{n+1} a_i, \\ i, j = 1, 2, \ldots, n\rangle.$$

By applying Tietze transformations, we have

$$\mathcal{H}(F_n) = \langle t, a_1, \ldots, a_n : (t, a_i, a_j) = 1 \ \forall i, j \in \{1, \ldots, n\}\rangle.$$

Let L be a free Lie algebra of rank $n+1$ with a free generating set $\{x_1, \ldots, x_{n+1}\}$, and let J be the ideal in L generated by the set

$$\mathcal{V} = \{[x_{n+1}, x_i, x_j] : i, j \in \{i, \ldots, n\}\}.$$

First, we show that J is a free Lie algebra, and it is a direct summand of L. Our proof is mainly based on the techniques developed in the paper "On the Mc-Cool group M_3 and its associated Lie algebra "by V. Metaftsis and A.I. Papistas (accepted in Commun. Algebra, 2016). Namely, let V_1 and V_2 be the $\mathbb{Z}$-modules spanned by the sets $\mathcal{V}_1 = \{x_1, \ldots, x_n\}$ and $\mathcal{V}_2 = \{x_{n+1}\}$, respectively. Since $\mathcal{V}_1 \cup \mathcal{V}_2$ is a free generating set of L, we have

$$L = L(V_1 \oplus V_2).$$

By the Lazard Elimination Theorem (LETh), we have

$$L = L(V_1) \oplus L(V_2) \oplus L(W),$$

where

$$W = \bigoplus_{m \geq 2} W_m$$

and, for $m \geq 2$,

$$W_m = \bigoplus_{\substack{a+b=m-2 \\ a,b \geq 0}} [V_2, V_1, {}_a V_1, {}_b V_2]$$

for all $m \geq 2$. Let

$$W^{(1)} = \bigoplus_{m \geq 2} W_m^{(1)},$$

where

$$W_m^{(1)} = [V_2, V_1, {}_{(m-2)} V_2]$$

and

$$W^{(2)} = \bigoplus_{m \geq 3} W_m^{(2)},$$

where

$$W_m^{(2)} = \bigoplus_{\substack{k+\gamma=m-1 \\ k \geq 2}} [V_2, {}_k V_1, {}_\gamma V_2].$$

We point out that

$$\begin{array}{rcl} L(W) & = & L(W^{(1)} \oplus W^{(2)}) \\ \text{(LETh)} & = & L(W^{(1)}) \oplus L(W^{(2)} \wr W^{(1)}) \end{array}$$

We prove that

$$J = L(W^{(2)} \wr W^{(1)})$$

(and so, J is a free Lie algebra). Therefore,

$$\begin{aligned} L &= L(V_1) \oplus L(V_2) \oplus L(W^{(1)}) \oplus L(W^{(2)} \wr W^{(1)}) \\ &= L(V_1) \oplus L(V_2) \oplus L(W^{(1)}) \oplus J. \end{aligned}$$

(That is, J is a direct summand of L and so, L/J is torsion-free $\mathbb{Z}$-module.) Furthermore, we prove that L/J is isomorphic to $\mathrm{gr}(\mathcal{H}(F_n))$ as a Lie algebra, by correcting a result of Cohen, Cohen and Prassidis.

Let $\mathrm{IA}(F_{n+1})$ be the group of IA-automorphisms of F_{n+1}. By choosing suitable IA-automorphisms of F_{n+1}, we construct a subgroup of $\mathrm{IA}(F_{n+1})$ isomorphic to $\mathcal{H}(F_n)$. This observation gives that $\mathrm{IA}(F_{n+1})$ is not linear group, which is a well known result. Since $\mathrm{IA}(F_{n+1})$ residually nilpotent, we obtain $\mathcal{H}(F_n)$ is residually nilpotent. Furthermore, since $\gamma_c(\mathcal{H}(F_n))/\gamma_{c+1}(\mathcal{H}(F_n))$ is torsion-free, we have $\mathcal{H}(F_n)$ is a Magnus group. Our next step is to give a formula for the rank of each $\gamma_c(\mathcal{H}(F_n))/\gamma_{c+1}(\mathcal{H}(F_n))$, and to examine whether or not $\mathrm{gr}(\mathcal{H}(F_n))$ is embedded into the Lie algebra of $\mathrm{IA}(F_{n+1})$.

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THE TRUE CIRCUIT CONJECTURE

CHRISTOS TATAKIS

Abstract. Let G be a finite connected graph on the vertex set $V(G)$ and on the edge set $E(G)$. With $I(G)$ we denote the toric ideal I_{A_G} in $\mathbb{K}[e_1,\dots,e_m]$, where $A_G = \{\alpha_e,\ e \in E(G)\} \subset \mathbb{Z}^n$. The set of the primitive binomials forms the Graver basis of I_{A_G} and is denoted by Gr_G. An irreducible binomial is called a circuit if it has minimal support. The set of the circuits is denoted by $\mathcal{C}_A$.

Consider any circuit $C \in \mathcal{C}_A$ and regard its support supp(C) as a subset of A. The lattice $\mathbb{Z}(\text{supp}(C)$ has finite index in the lattice $\mathbb{R}(\text{supp}(C) \cap \mathbb{Z}A$, which is called the index of the circuit C and denoted by index(C). The true degree of the circuit C is the product $\deg(C)\cdot$ index(C).

B. Sturmfels in his lecture at Santa Cruz (July 1995), made the conjecture that circuits always have the maximal degree among the elements of the Graver basis. Hosten and R. Thomas gave a counterexample of a toric ideal such that the maximal degree of the elements of the Graver basis was 16 while the maximal degree of the circuits was 15. This example led B. Sturmfels to alter the conjecture to the following:

True circuit conjecture (Sturmfels, 1996): Let t_A be the maximal true degree of any circuit in $\mathcal{C}_A$. Then it holds that

$$\deg(B) \leq t_A,$$

for every element B in the Graver basis Gr_A of a toric ideal I_A.

We present an infinite family of counterexamples to the true circuit conjecture by providing toric ideals and elements of the Graver basis for which their degrees are not bounded above by t_A. Next, we extend this result, by studying the following question:

Question: Does the degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by a constant times $(t_A)^2$ or a constant times $10^{2016}(t_A)^{2016}$?

We provide a family of graphs for which the degree of any element in the Graver basis of the corresponding toric ideal cannot be bounded polynomially above by the maximal true degree of any circuit.

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THE RICCI FLOW IN DIMENSION THREE

ILIAS TERGIAKIDIS

ABSTRACT. Let (M, g) be a smooth, closed Riemannian manifold. The Ricci flow

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2\mathrm{Ric}(g(t)) \\ g(0) &= g_0 \end{aligned}$$

is a PDE that evolves the metric tensor. In this talk we will make an introduction to some special solutions to the Ricci flow, namely the Ricci solitons. The Ricci solitons can be regarded as generalized fixed points of the flow and they correspond to those solutions, which change only by a diffeomorphism and rescaling under the Ricci flow.

1. INTRODUCTION

Let M be a smooth, closed (i.e. compact and without boundary) manifold with Riemannian metric $g_0 \in \Gamma(S^2_+ T^* M)$. The Ricci flow is a PDE that evolves the metric tensor:

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2\mathrm{Ric} \\ g(0) &= g_0, \end{aligned}$$

where $\{g(t)\}$ is a one-parameter family of metrics on M and $\mathrm{Ric} := \mathrm{Ric}(g(t))$ denotes the Ricci curvature. It will be clear later that the minus sign makes the Ricci flow a heat-type equation, so it is expected to "average out" the curvature.

In order to get a feel for the evolution equation, we will look at some simple examples.

Example 1.1 (Einstein metrics)**.** *Suppose that the initial metric g_0 is Ricci flat, i.e. $Ric(g_0) = 0$. In this case the metric will remain stationary for all subsequent times. Concrete examples are the Euclidean space $\mathbb{R}^n$ and the flat torus $\mathbb{T}^n = S^1 \times ... \times S^1$. Suppose now that the initial metric is an Einstein metric, i.e. $Ric(g_0) = \lambda g_0$, $\lambda \in \mathbb{R}$. A solution $g(t)$ with $g(0) = g_0$ is given by*

$$g(t) = (1 - 2\lambda t) g_0.$$

If $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ we call the solution shrinking, steady or expanding respectively. The simplest shrinking solution is that of the unit sphere $(\mathbb{S}^n, g_0)$ endowed with the round metric. It holds that $Ric(g_0) = (n-1)g_0$, so $g(t) =$

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$(1-2(n-1)t)g_0$ is a solution to the Ricci flow defined on the maximal time interval $(-\infty, T)$, where $T = \frac{1}{2(n-1)}$. That is, under the Ricci flow $\mathbb{S}^n$ stays round and shrinks homothetically at a steady rate. Observe that at time T the sphere shrinks to a point. By contrast, the simplest expanding solution is that of the hyperbolic space $\mathbb{H}^n$ endowed with the hyperbolic metric (constant sectional curvature -1). In this case $Ric(g_0) = -(n-1)g_0$, so $g(t) = (1+2(n-1)t)g_0$ is a solution to the Ricci flow and the manifold expands homothetically for all time.

Example 1.2 (Quotient metrics)**.** *Let $M = N/G$ be a quotient of a Riemannian manifold N by a discrete group of isometries G. Then it will remain so under the Ricci flow, as the Ricci flow on N preserves the isometry group. For example $\mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2$ shrinks to a point in finite time, as does its cover $\mathbb{S}^n$.*

Example 1.3 (Product Metrics)**.** *Let $M \times N$ be a product manifolds endowed with the product metric $g_M \oplus g_N$. Under the Ricci flow the metric will remain a product metric and its factor evolves independently. For example for $S^2 \times S^1$, the first factor shrinks to a point in finite time, while the second factor stays stationary.*

2. Short Time Existence and Uniqueness

2.1. **Existence Theory for Parabolic PDEs.** Let M be a Riemannian manifold and $\pi : E \to M$ a vector bundle over M. Consider now a (time-dependent) section $u : M \times [0, T) \to E$ given locally for some local frame (e_k) by $u = u^k e_k$. We are interested in PDEs describing the evolution of u:

$$\begin{aligned} \frac{\partial u}{\partial t} &= L(u) \\ u(x, 0) &= u_0(x), \end{aligned}$$

where $L : \Gamma(E) \to \Gamma(E)$ is some second-order differential operator given in terms of local coordinates (x^i) on M and the local frame (e_k) on E as

$$L(u) = \Big((\lambda_{ij})^k_l \frac{\partial^2 u^l}{\partial x^i \partial x^j} + (\mu_i)^k_l \frac{\partial u^l}{\partial x^i} + \nu^k_l u^l \Big) e_k \tag{2.1}$$

where, $\lambda \in \Gamma(S^2 T^* M \otimes \mathrm{Hom}(E, E))$, $\mu \in \Gamma(T^* M \otimes \mathrm{Hom}(E, E))$ and $\nu \in \Gamma(\mathrm{Hom}(E, E))$.

Definition 2.1. *Let $\zeta \in \Gamma(T^* M)$. The total symbol of L in the direction ζ is the bundle homomorphism*

$$(\sigma[L](\zeta))(u) = ((\lambda_{ij})^k_l \zeta^i \zeta^j u^l + (\mu_i) l^k \zeta^i u^l + \lambda^k_l u^l) e_k.$$

The principal symbol of L in the direction ζ is now the bundle homomorphism of only the highest order terms, that is

$$\hat{\sigma}[L](\zeta) = (\lambda_{ij})^k_l \zeta^i \zeta^j (e^*)^l \otimes e_k.$$

Remark 2.2. *As Hamilton noted in* [4], *computing the symbol is easily obtained (at least heuristically) by replacing the derivatives $\frac{\partial}{\partial x_i}$ by the Fourier transformation variable ζ_i.*

Definition 2.3. *L is called elliptic if its principal symbol $\hat{\sigma}[L](\zeta)$ is an isomorphism whenever $\zeta \neq 0$.*

Definition 2.4. *The system*

$$\frac{\partial u}{\partial t} = L(u)$$

$$u(x,0) = u_0(x),$$

is called strongly parabolic if there exists $\delta > 0$, such that at each point of M, for all $\phi \neq 0$ and $u \neq 0$

$$\langle \hat{\sigma}[L](\zeta)(u), u \rangle > \delta |\zeta|^2 |u|^2.$$

The definition implies in particular, that the principal symbol in any direction is a linear isomorphism of the fibre. A linear equation of the form $\partial_t u = Lu$ is parabolic if L is elliptic.

When faced with a non-linear PDE, one attempts to linearise the equation in such a way that linear theory can be applied. We are specifically interested in the linearisation of a non-linear operator L on a vector bundle E.

Definition 2.5. *If $u(t) \in \Gamma(E)$ is a time-dependent section of E, such that $u(0) = u_0$ and $u'(0) = v$, then the linearization of L at u_0 is the linear map*

$$\begin{aligned} DL : \Gamma(E) &\to \Gamma(E) \\ v &\mapsto \frac{\partial}{\partial t} L(u(t))|_{t=0}. \end{aligned}$$

Definition 2.6. *The system*

$$\begin{aligned} \frac{\partial u}{\partial t} &= L(u) \\ u(x,0) &= u_0(x), \end{aligned} \tag{2.2}$$

is strongly parabolic at u_0 if the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= DL(u) \\ u(x,0) &= u_0(x). \end{aligned}$$

is parabolic in the sense described above.

Theorem 2.7. *If the system (2.2) is strongly parabolic at u_0, then there exists a solution on some time interval $[0,T)$ and the solution is unique for as long as it exists.*

2.2. **The DeTurck trick.** A crucial step in the study of geometric evolution equations is to show short time existence and uniqueness. As Theorem 2.7 states, the system must be strongly parabolic. We will show that the Ricci flow fails to be strongly parabolic. However Hamilton managed to overcome this difficulty by using the Nash-Moser Implicit Function Theorem. A little time later DeTurck in [3] found a more direct proof by modifying the flow by a time-dependent change of variables to make it parabolic. This is also the method we will follow in this survey.

We would like to regard the Ricci tensor as a nonlinear operator on the space of metrics, i.e. $\mathrm{Ric} : \Gamma(S^2_+ T^*M) \to \Gamma(S^2 T^*M)$. We define the linearisation of the Ricci tensor

$$(D\mathrm{Ric})\Big(\frac{\partial g_{ij}}{\partial t}\Big) = \frac{\partial}{\partial t} \mathrm{Ric}(g_{ij}(t))|_{t=0}.$$

Lemma 2.8. *The linearisation of the Ricci tensor is given by*

$$(DRic)(h)_{jk} = \frac{1}{2} g^{pq} (\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp}).$$

Proof. The proof is immediate if we use the following fact. If $\frac{\partial}{\partial t} g_{ij} = h_{ij}$, where h is some symmetric 2-tensor, then the variation formula for the Ricci tensor is given by

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{pq} (\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp}).$$

A proof of this statement can be found in [1] pg. 69, [2] pg. 109 and [7] pg.22. □

In order to check if the system is strongly parabolic we must calculate its principal symbol. The principal symbol in the direction ζ of the linear operator $D(-2\text{Ric})$ (as a function of the metric g) is the bundle homomorphism

$$\hat{\sigma}[-2D\text{Ric}](\zeta) : S^2_+(T^*M) \to S^2(T^*M)$$

and is obtained by replacing the covariant derivative ∇_i by the covector ζ_i, namely

$$\hat{\sigma}[-2D\text{Ric}](\zeta)(h)_{jk} = g^{pq}(-\zeta_q \zeta_j h_{kp} - \zeta_q \zeta_k h_{jp} + \zeta_q \zeta_p h_{jk} + \zeta_j \zeta_k h_{qp}).$$

Now the Ricci flow is strongly parabolic if there exists some $\delta > 0$, such that for all $\zeta \neq 0$ and all $h_{ij} \neq 0$

$$\langle \hat{\sigma}[-2D\text{Ric}](\zeta)(h), h \rangle > \delta |\zeta|^2 |h|^2,$$

which can be rewritten as

$$g^{pq}(-\zeta_q \zeta_j h_{kp} - \zeta_q \zeta_k h_{jp} + \zeta_q \zeta_p h_{jk} + \zeta_j \zeta_k h_{qp}) h^{ij} > \delta \zeta_i \zeta^i h_{rs} h^{rs}.$$

Observe that if we choose $h_{kp} = \zeta_k \zeta_p$, then the left hand side is zero. Therefore the Ricci flow in not strongly parabolic.

Remark 2.9. *The fact that the principal symbol has non-trivial kernel is related to the invariance of the Ricci tensor under diffeomorphism,*

$$Ric(\phi^* g(t)) = \phi^* Ric(g(t)).$$

See [1], *Section 2.2 for more details.*

Because the Ricci flow in not strongly parabolic we can not apply Theorem 2.7 immediately. The construction we will describe is due to DeTurck. He showed that it is possible to modify the Ricci flow and thereby obtain a parabolic PDE by a clever trick: one modifies the right-hand side of the equation by adding a term which is a Lie derivative of the metric with respect to a certain vector field which in turn depends on the metric. Remarkably, one then can obtain a solution to the original Ricci flow equation by pulling back the solution of the modified flow by appropriately chosen diffeomorphisms.

In order to motivate DeTurck's idea we will rewrite the linearisation of -2Ric, to see which terms are causing the weak parabolicity.

$$(-2D\text{Ric})(h)_{jk} = g^{qp}(\nabla_q \nabla_p h_{jk} + \nabla_j \nabla_k h_{qp} - \nabla_q \nabla_j h_{kp} - \nabla_q \nabla_k h_{jp}),$$

because $g^{qp} = g^{pq}$. Then

$$(-2D\text{Ric})(h)_{jk} = \Delta h_{jk} + g^{qp}(\nabla_j \nabla_k h_{qp} - \nabla_q \nabla_j h_{kp} - \nabla_q \nabla_k h_{jp}).$$

By the formula for commuting covariant derivatives (look at [2] pg. 14) we have that

$$\begin{aligned} \nabla_q \nabla_j h_{kp} &= \nabla_j \nabla_q h_{kp} - R^r_{qjk} h_{rp} - R^m_{qjp} h_{km} \\ &= \nabla_j \nabla_q h_{kp} + \text{ lower order terms in } h. \end{aligned}$$

So by using that $\nabla g = 0$ we obtain

$$\begin{aligned}(-2D\mathrm{Ric})(h)_{jk} &= \Delta h_{jk} + \nabla_j\big[g^{pq}(\frac{1}{2}\nabla_k h_{pq} - \nabla_q h_{kp})\big] + \nabla_k\big[g^{pq}(\frac{1}{2}\nabla_j h_{pq} - \nabla_q h_{jp})\big] \\ &\quad + \text{lower order terms in } h \\ &= \Delta h_{jk} + \nabla_j V_k + \nabla_k V_j + \text{ lower order terms in } h.\end{aligned}$$

The lower order terms have no contribution to the principal symbol. The first term is a good term, but the terms in V are bad and make the Ricci flow non-parabolic. Furthermore observe that $\nabla_j V_k + \nabla_k V_j$ corresponds to a Lie derivative term.

By the discussion above V_j is defined by

$$\begin{aligned}V_j &= g^{pq}(\frac{1}{2}\nabla_j h_{pq} - \nabla_q h_{qj}) \\ &= -\frac{1}{2}g^{pq}(\nabla_p h_{qj} + \nabla_q h_{pj} - \nabla_j h_{pq}).\end{aligned}$$

Remark 2.10. *If* $\frac{\partial}{\partial t} g_{ij} = h_{ij}$, *where* h *is some symmetric* 2*-tensor, then the variation formula for the Christoffel symbols is given by*

$$\frac{\partial}{\partial t}\Gamma^i_{pq} = \frac{1}{2}g^{ij}(\nabla_p h_{qj} + \nabla_q h_{pj} - \nabla_j h_{pq}).$$

A proof of this statement can be found in [1] *pg. 68,* [2] *pg. 108 and* [7] *pg.20.*

Let

$$D\Gamma_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M \otimes TM)$$

denote the linearisation of the Levi-Civita connection and it is given by

$$((D\Gamma_g)(h))^i_{pq} = \frac{\partial}{\partial t}\Gamma^i_{pq}|_{t=0}.$$

Then

$$V_j = -g^{pq}g_{ij}((D\Gamma_g)(h))^i_{pq}.$$

We wish to add an appropriate correction term to the Ricci tensor to make it elliptic. Fix a background metric $\tilde{g}$ on M with Levi-Civita connection $\tilde{\Gamma}$. The considerations above lead us to define a vector field W by

$$W^i = g^{pq}(\Gamma^i_{pq} - \tilde{\Gamma}^i_{pq}).$$

As the difference of two connections is a tensor, it is a globally well defined vector field. But W involves only one derivative of g, so the map

$$\begin{aligned}Q(\tilde{\Gamma}) : \Gamma(S^2T^*M) &\to \Gamma(S^2T^*M) \\ g &\mapsto \mathcal{L}_g W\end{aligned}$$

corresponds to a second order differential operator. The linearisation of Q is given by

$$(DQ)(h)_{jk} = \nabla_j V_k + \nabla_k V_j + \text{ lower order terms in } h.$$

This leads us to define a modified Ricci operator

$$P(g(t)) = -2\mathrm{Ric}(g(t)) - Q(g(t)) = -2\mathrm{Ric}(g(t)) - \mathcal{L}_W g(t).$$

The linearisation of P is thus given by

$$(DP)(h) = \Delta h + \text{ lower order terms in } h,$$

hence the principal symbol is given by

$$\hat{\sigma}[DP](\zeta)(h) = |\zeta|^2 h. \tag{2.3}$$

DeTurck demonstrated a strategy for constructing a unique short-time solution $\bar{g}(t)$ of the Ricci flow

$$\begin{aligned} \frac{\partial}{\partial t}\bar{g}(t) &= -2\mathrm{Ric}(\bar{g}(t)) \\ \bar{g}(0) &= g_0 \end{aligned}$$

on a closed manifold M. We will give an outline (further details in [1] pg. 80,81):

- The Ricci-DeTurck flow is given by

$$\begin{aligned} \frac{\partial}{\partial t}g_{ij} &= -2R_{ij} + \nabla_i W_j + \nabla_j W_i \\ g(0) &= g_0, \end{aligned}$$

  where $W_j = g_{jk}W^k = g_{jk}g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq})$. It follows from (2.3), that the system is strongly parabolic, thus by Theorem 2.7 short time existence and uniqueness hold.
- One observes that the one pararameter family of vector fields $W(t)$ exist as long as the solution $g(t)$ exists. Then one defines a one parameter family of diffeomorphisms $\phi_t : M \to M$ by

$$\begin{aligned} \frac{\partial}{\partial t}\phi_t &= W(t) \\ \phi_0 &= \mathrm{id}_M. \end{aligned}$$

  But M is compact, so from ([1], Lemma 3.15, pg. 82) ϕ_t exist and remain diffeomoprhisms for as long as the solution $g(t)$ exists, namely for $t \in [0, T)$.
- The family of metrics $\bar{g}(t) := \phi_t^* g(t)$ defined for $0 \leq t < T$ is a solution to the Ricci flow.
- For uniqueness it suffices to prove that a solution to Ricci- DeTurck flow is produced from a solution of Ricci flow after a reparametrization defined for harmonic map heat flow. We won't give this technical proof, but the interested reader could look up [1] Chapter 3.4.

Finally we can state the short time existence and uniqueness result for the Ricci flow.

Theorem 2.11 (Short Time Existence and Uniqueness for the Ricci Flow)**.** *Given a smooth metric g_0 on a closed manifold M, there exists a maximal interval $[0, T)$, such that a solution $g(t)$ of the Ricci flow with $g(0) = g_0$ exists, is smooth on $[0, T)$ and this solution is unique.*

3. The Maximum Principle

The maximum principle is the main tool we will use to understand the behaviour of solutions to the Ricci flow. It is a very powerful tool which can be used to show that pointwise inequalities on the initial data of parabolic PDEs are preserved under the evolution. The question of what it means for a tensor quantity to "average out" naturally arises.

3.1. **The Scalar Maximum Principle.**

Theorem 3.1 (Weak Maximum Principle for Scalars)**.** *Let M be a closed manifold. Suppose that for $t \in [0,T)$, $g(t)$ is a one-parameter family of metrics on M and $X(t)$ a one-parameter family of vector fields on M. Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Furthermore assume that $u : M \times [0,T) \to \mathbb{R}$ is a C^2 solution to*

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u).$$

Such kind of solutions are called subsolutions. Now suppose that there exists $C \in \mathbb{R}$, such that $u(x,0) \leq C$ for all $x \in M$ and let $\phi : [0,T) \to \mathbb{R}$ be a solution to the associated ODE

$$\begin{aligned} \frac{d\phi}{dt} &= F(\phi) \\ \phi(0) &= C. \end{aligned}$$

Then $u(x,t) \leq \phi(t)$ for all $x \in M$ and $t \in [0,T)$ in the interval of existence of ϕ.

Analogous, if

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u)$$

the solution is called a supersolution. In this case under the assumptions that $u(x,0) \geq C$ for all $x \in M$ and ϕ is a solution to the associated PDE, we obtain that $u(x,t) \geq \phi(t)$ for all $x \in M$ and $t \in [0,T)$ in the interval of existence of ϕ.

Proof. The proof can be found in [1] pg. 96. □

3.2. **The Maximum Principle for Vector Bundles.**

Let M be a closed oriented manifold equiped with a smooth one parameter family of metrics $g(t)$, $t \in [0,T)$ and their Levi-Civita connections $\nabla(t)$. Let $\pi : E \to M$ be a vector bundle over M with a fixed bundle metric h. Let

$$\tilde{\nabla}(t) : \Gamma(E) \to \Gamma(E \otimes T^*M)$$

be a smooth family of connections compatible with h in the sense that for all $X \in \Gamma(TM)$, sections $\phi, \psi \in \Gamma(E)$ and times $t \in [0,T)$ one has

$$X(h(\phi,\psi)) = h(\tilde{\nabla}_X \phi, \psi) + h(\phi, \tilde{\nabla}_X \psi).$$

In order to define the Laplacian of a section ϕ we have to take two covariant derivatives of ϕ and then take the trace. The first covariant derivative is $\tilde{\nabla}\phi \in \Gamma(E \otimes T^*M)$ and we encounter a problem, because $\tilde{\nabla}\phi$ is not a section of E, so we cannot simply take the second covariant derivative using $\tilde{\nabla}(t)$. We need to resolve this situation. We define a new connection

$$\hat{\nabla}(t) : \Gamma(E \otimes T^*M) \to \Gamma(E \otimes T^*M \otimes T^*M)$$

for all $X \in \Gamma(TM)$, $\phi \in \Gamma(E)$ and $\xi \in \Gamma(T^*M)$ by

$$\hat{\nabla}_X(\phi \otimes \xi) := \tilde{\nabla}_X \phi \otimes \xi + \phi \otimes \nabla_X \xi.$$

Then the time dependent bundle Laplacian $\hat{\Delta}(t)$ is defined for all $\phi \in \Gamma(E)$ as the metric trace

$$\hat{\Delta}\phi := \mathrm{tr}_g \hat{\nabla}(\tilde{\nabla}\phi).$$

We are going to state now a maximum principle for sections of a vector bundle. In the scalar maximum principle case we showed, that the lower bound of the solution is presevered. Now we will show, that in the case of vector bundles the solution stays inside convex sets. The set should also be closed and invariant under parallel translation.

Theorem 3.2 (Maximum Principle for Vector Bundles). *Let $F : E \times [0,T) \to E$ be a continuous map such that $F(\cdot,\cdot,t) : E \to E$ is fiber preserving for each $t \in [0,T)$ and $F(\cdot,x,t) : E_x \to E_x$ is Lipschitz for all $x \in M$ and $t \in [0,T)$. Let K be a closed subset of E such that:*

- *K is invariant under parallel translation by $\tilde{\nabla}(t)$ for all $t \in [0,T)$ and*
- *$K_x = K \cap \pi^{-1}(x)$ is a closed convex subset of $E_x = \pi^{-1}(x)$ for all $x \in M$.*

Furthermore assume that $\alpha(t)$, $t \in [0,T)$ is a time dependent section of E that is a solution of the nonlinear PDE

$$\frac{\partial \alpha}{\partial t} = \hat{\Delta}\alpha + F(\alpha),$$

such that $\alpha(0) \in K$. Suppose now that every solution of the ODE

$$\begin{aligned} \frac{d\alpha}{dt} &= F(\alpha) \\ \alpha(0) &\in K_x \end{aligned}$$

remains in K_x. Then the solution $\alpha(t)$ of the PDE remains in K.

4. Derivative Estimates and Curvate Blow-Up at Singularities

4.1. **Evolution of Geometric Quantities under the Ricci Flow.** In order to apply maximum principle arguments to the curvature, we need to know how curvature quantities evolve under the Ricci flow.

Remark 4.1. *Recall the following local expressions, which can be found in any Riemannian geometry book.*
The components of the (4,0) Riemann curvature tensor:

$$R_{ijkl} = g_{lp} R^p_{ijk}.$$

The symmetries of the Riemann curvature tensor:

$$R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijkl}.$$

Observe that the symmetries above allow us to view the Riemann curvature tensor as a section $Rm \in \Gamma(S^2(\Lambda^2 T^*M))$. The first Bianchi identity:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0.,$$

The second Bianchi identity:

$$\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0.$$

The components of the Ricci curvature tensor:

$$R_{ij} = R^p_{pij} = g^{lp} R_{pijl},$$

with the symmetry $R_{ij} = R_{ji}$.
The scalar curvature:

$$R = R^i_i = g^{ij} R_{ij}.$$

The contracted second Bianchi identity:

$$\nabla^j R_{ij} = \frac{1}{2} \nabla_i R.$$

Finally the (rough) Laplacian is a family of operators $\Delta : \Gamma(\mathcal{T}_l^k M) \to \mathcal{T}_l^k M)$ *defined by:*

$$\Delta F := g^{ij} \nabla_i \nabla_j F.$$

Lemma 4.2. *Suppose that* $g_{ij}(t)$ *is a solution of the Ricci flow:*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

Then the various geometric quantities evolve according to the following equations:

(1) *Metric inverse:*

$$\frac{\partial}{\partial t} g^{ij} = 2R^{ij}$$

(2) *Christoffel Symbols:*

$$\frac{\partial}{\partial t} \Gamma_{ij}^k = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij})$$

(3) *(4,0)-Riemann curvature tensor:*

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk}) \\ &\quad -(R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}), \end{aligned}$$

where $B_{ijkl} = -R_{pij}^q R_{qlk}^p$.

(4) *Ricci tensor:*

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2g^{pq} g^{rs} R_{pijr} R_{qs} - 2g^{pq} R_{ip} R_{qj}.$$

(5) *Scalar Curvature:*

$$\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2.$$

Proof. One should be at first aware of the variation formulas of geometric quantities under the equation $\frac{\partial}{\partial t} g_{ij} = h_{ij}$, where h is some symmetric 2-tensor. These formulas can be found in [1] pg. 68, [2] pg. 108 and [7] pg.20. Most of the equation above follow from these results by applying $h = -2\mathrm{Ric}$ with some extra work. These results can be found in [1] pg. 174-179, [2] pg. 108-113 and [7] pg.32-34. In particular the proofs of the evolution equations for Rm, Ric and R are lengthy and need some extra work. One has to apply the formula for commuting covariant derivatives (which can be found in [2] pg. 14, [6] pg. 94) and afterwards use the Bianchi identities. □

4.2. **Evolution of the Derivatives of Curvature.** We will look at the square of the norm of the derivatives of the Riemann curvature, i.e. $|\nabla^k \mathrm{Rm}|^2$, where k denotes the k-th iterated covariant derivative. The computations in this section will be crucial for the next one. We will see in the next section that the Bernstein-Bando-Shi estimates give us upper bounds for $|\nabla^k \mathrm{Rm}|^2$. In order to prove them we should apply the maximum principle of Theorem 3.1 for the evolution equations $\frac{\partial}{\partial t} |\nabla^k \mathrm{Rm}|^2$. Our aim is to prove the following proposition:

Proposition 4.3. *The square of the norm of the k-th covariant derivative of the Riemann curvature tensor satisfies the heat-type equation*

$$\frac{\partial}{\partial t} |\nabla^k Rm|^2 = \Delta |\nabla^k Rm|^2 - 2|\nabla^{k+1} Rm|^2 + \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j} Rm * \nabla^k Rm.$$

Remark 4.4. *We adopt the following convention: If A and B are two tensors on a n-dimensional Riemannian manifold, we denote by $A * B$ any quantity obtained from $A \otimes B$ by one or more of these operations: (1) summation over pairs of matching upper nad lower indices, (2) contraction on upper indices with respect to the metric, (3) contraction on lower indices with respect to the metric inverse and (4) multiplication by constants depending only on n and the ranks of A and B. We also denote by A^{*k} any k-fold product $A * ... * A$.*

The proof of the previous proposition is given in [1] Chapter 7.2 and [7] Chapter 3.3 as a part of the proof of the Bernsten-Bando-Shi estimates. We prove it in this survey sepperately as suggested in [6]. In order to give the proof of the proposition we will need the following three lemmas.

Lemma 4.5. *Let A, F be two tensors of the same type. If A satisfies*

$$\frac{\partial}{\partial t}A = \Delta A + F$$

under the Ricci flow, then

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + F * A + Ric * A^{*2}.$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}g_t(A, A) &= 2g_t(\frac{\partial}{\partial t}A, A) + \frac{\partial g(t)}{\partial t}(A, A) \\
&= 2g_t(\Delta A + F, A) + \mathrm{Ric} * A^{*2} \\
&= \Delta|A|^2 - 2|\nabla A|^2 + F * A + \mathrm{Ric} * A^{*2}.
\end{aligned}$$

We have used the identity

$$\Delta|A|^2 = 2\langle \Delta A, A\rangle + 2|\nabla A|^2.$$

□

Lemma 4.6. *Let A, F be two tensors of the same type. If A satisfies*

$$\frac{\partial}{\partial t}A = \Delta A + F$$

under the Ricci flow, then

$$\frac{\partial}{\partial t}\nabla A = \Delta(\nabla A) + \nabla F + Rm * \nabla A + \nabla Ric * A.$$

Proof. ∇A can be written as

$$\nabla A = \partial A + f(\Gamma, A),$$

where $f(\Gamma, A)$ is some expression of the form $\Gamma * A$. ([2] pg. 8, [6] pg. 11 give the coordinate expression of ∇A for a tensor field A). Further more by Lemma 4.2, (2)

$$\partial_t \Gamma = (g^{-1}) * \nabla \mathrm{Ric}.$$

Now

$$\begin{aligned}
\partial_t \nabla A &= \partial_t \partial A + \partial_t f(\Gamma, A) \\
&= \partial \partial_t A + f(\Gamma, \partial_t A) + f(\partial_t \Gamma, A) \text{ (by the product rule)} \\
&= \nabla(\partial_t A) + f(g^{-1} * \nabla \mathrm{Ric}, A) \text{ (because } \partial_t A \text{ is a tensor of the same type as } A) \\
&= \nabla(\Delta A + F) + \nabla \mathrm{Ric} * A
\end{aligned}$$

$$\begin{aligned} &= (\Delta\nabla A + \mathrm{Rm} * \nabla A + \nabla \mathrm{Ric} * A) + \nabla F + \nabla \mathrm{Ric} * A \\ &= \Delta\nabla A + \nabla F + \mathrm{Rm} * \nabla A + \nabla \mathrm{Ric} * A. \end{aligned}$$

We used the formula

$$[\nabla, \Delta]A := \nabla\Delta A - \Delta\nabla A = \mathrm{Rm} * \nabla A + \nabla \mathrm{Ric} * A,$$

which folows from the formula for commuting covariant derivatives (which can be found in [2] pg. 14, [6] pg. 94) followed by the second Bianchi identity (more details in [2] pg. 227). ☐

Observe that the formula for the evolution of the Riemann curvature tensor in Lemma 4.2, (3) can be written as

$$\frac{\partial}{\partial t}\mathrm{Rm} = \Delta \mathrm{Rm} + \mathrm{Rm}^{*2}. \tag{4.1}$$

Lemma 4.7. *We have the following evolution equation under the Ricci flow:*

$$\frac{\partial}{\partial t}\nabla^k Rm = \Delta\nabla^k Rm + \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j} Rm.$$

Proof. The proof uses induction. The case $k = 0$ corresponds to the evolution equation given by (4.1). We assume that it holds for k and we apply the previous Lemma for $A = \nabla^k \mathrm{Rm}$ and

$$F = \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm}.$$

The statement of the Lemma is straightforward if we observe that all of the reaction terms on the RHS are of the form $\nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm}$, where $i + j = k + 1$. ☐

We are now in position to prove Proposition 4.3.

Proof. We simply apply the result of the previous Lemma to the first Lemma. Then the statement of the Proposition is straightforward if we observe that all the terms on the LHS other than the first two are of the form $\nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^k \mathrm{Rm}$ where $i + j = k$. ☐

4.3. **The Global Derivative Estimates.** The Global Derivative Estimates, known also as Bernstein-Bando-Shi Estimates will be obtained by applying the maximum principle of Theorem 3.1 to the evolution equations derived in Proposition 4.3. In this way we will obtain bounds on the derivatives of the curvature. However there are some problems when we try to apply the maximum principle to the evolution equation of the previous chapter. The first problem is that we cannot guarantee any initial conditions on the derivatives of the curvature if we are only given bounds on the curvature. The second problem is that the evolution equation has some terms in it, which we are not sure how to control. The statement and the proof of the Bernstein-Bando-Shi estimates can be found in [1] Chapters 7.1 and 7.2, [7] Chapter 3.3 and [6] Chapter 6.3. We will demonstrate here an outline of the proof.

Theorem 4.8 (The Bernstein-Bando-Shi Estimates)**.** *Let M^n be a closed manifold and $g(t)$ a smooth solution of the Ricci flow. Then for each $\alpha > 0$ and $m \in \mathbb{N}$, there exists a constant C_m depending only on m, n and $max\{\alpha, 1\}$ such that if*

$$|Rm(x,t)|_{g(x,t)} \leq K, \ \textit{for all } t \in [0, \frac{\alpha}{K}],$$

then

$$|\nabla^m Rm(x,t)|_{g(x,t)} \leq \frac{C_m K}{t^{m/2}}, \text{ for all } t \in (0, \frac{\alpha}{K}].$$

Proof. We will demonstrate an outline of the proof. The result can be obtained by induction on m.

- Observe that for $m = 0$ the result is truy by hypethesis, with $C_0 = 1$. Assume that it holds for $p \leq m-1$.
- By the result of Proposition 4.3 and the inductive hypothesis we can show that
$$\frac{\partial}{\partial t}|\nabla^m \mathrm{Rm}|^2 \leq \Delta|\nabla^m \mathrm{Rm}|^2 - 2|\nabla^{m+1}\mathrm{Rm}|^2 + \bar{C}_m K\Big(|\nabla^m \mathrm{Rm}|^2 + \frac{K^2}{t^m}\Big)$$
for $t \in (0, \frac{\alpha}{K}]$, where $\bar{C}_m^{'}$ is a constant depending only on m and n.
- In order to get the desired bound we are seeking for an upper bound on $t^m|\nabla^m \mathrm{Rm}|^2$. At $t = 0$ this quantity equals zero, so it does have an upper bound for $t = 0$. This resolves the first problem. The evolution equation in this case is given by

$$\frac{\partial}{\partial t}(t^m|\nabla^m \mathrm{Rm}|^2) \leq \Delta|t^m \nabla^m \mathrm{Rm}|^2 - 2t^m|\nabla^{m+1}\mathrm{Rm}|^2 + (\bar{C}_m K t + m)t^{m-1}|\nabla^m \mathrm{Rm}|^2 + \bar{C}_m K^3.$$

In order to apply the maximum principle of Theorem 3.1 and get an upper bound, we need to show that the reaction terms in the evolution equation cause $t^m|\nabla^m \mathrm{Rm}|^2$ to decrease. Unfortunately the reaction terms (i.e. the last two terms in the evolution equation above) are not negative. This is the second problem.

- To fix this problem the proof becomes technical. The idea is that we make use of the term $-2|\nabla^{k+1}\mathrm{Rm}|^2$ in the evolution equation of Proposition 4.3. By adding the right amount of $t^{m-1}|\nabla^{m-1}\mathrm{Rm}|^2$ (which we know by the inductive hypothesis is bounded above by a constant) we can cancel off the unruly reactionary terms involving $t^m|\nabla^m \mathrm{Rm}|^2$. In so doing we will introduce new unruly terms in $t^{m-1}|\nabla^{m-1}\mathrm{Rm}|^2$, so we will need to add the right amount of the next derivative down and so on. We define
$$G := t^m|\nabla^m \mathrm{Rm}|^2 + \sum_{j=0}^{m-1} \alpha_{mj} t^j |\nabla^j \mathrm{Rm}|^2.$$
- One can show that
$$\frac{\partial}{\partial t}G \leq \Delta G + B_m K^3,$$
for $B_m := \bar{C}_m + \sum_{j=0}^{m-1} \alpha_{mj} D_j$, where D_j are numbers depending only on j, n for $1 \leq j \leq m-1$.
- The reaction term is simply a constant, so it gives linear growth at worst. At $t = 0$, $G = \alpha_{m0}|\mathrm{Rm}|^2 \leq \alpha_{m0}K^2$. Now by applying the maximum principle of Theorem 3.1 we obtain
$$G \leq \alpha_{m0}K^2 + B_m K^3 t \leq (\alpha_{m0} + B_m \alpha)K^2 := C_m^2 K^2,$$
for $t \in [0, \frac{\alpha}{K})$, where C_m is a constant depending only on m, n and $\max\{\alpha, 1\}$. Finally
$$|\nabla^m \mathrm{Rm}| \leq \sqrt{\frac{G}{t^m}} \leq \frac{C_m K}{t^{m/2}},$$

for $t \in (0, \frac{\alpha}{K}]$.

□

4.4. **Curvature Blows-up at Finite-time Singularities.** We are going to combine the short time existence result for the Ricci flow (Theorem 2.11) and the global derivative estimates of Bernstein-Bando-Shi (Theorem 4.8) in order show that if the Ricci flow becomes singular, then the curvature blows-up as we approach the singular time. In particular the theorem we aim to demonstrate in this section is the following:

Theorem 4.9 (Curvature blows-up at a singularity)**.** *If g_0 is a smooth metric on a compact manifold M, the Ricci flow with $g(0) = g_0$ has a unique solution $g(t)$ on a maximal time interval $t \in [0, T)$. Moreover, if $T < \infty$, then*

$$\lim_{t \to T} \left(\sup_{x \in M} |Rm(x,t)| \right) = \infty.$$

Before we are in position to prove this theorem, we shall need some key results. The first one is the following:

Lemma 4.10. *Let M be a closed manifold and $g(t)$ with $t \in [0, T)$ a one-parameter family of metrics on M depending smoothly on space and time. If there exists a constant $C < \infty$ such that*

$$\int_0^T \left| \frac{\partial}{\partial t} g(t,x) \right|_{g(t)} dt \leq C,$$

for all $x \in M$, then the metric $g(t)$ converges uniformly to a continuous metric $g(T)$ such that for all $x \in M$

$$e^{-C} g(x,0) \leq g(x,T) \leq e^{C} g(x,0).$$

Proof. The proof can be found in [1] pg. 203. □

Remark 4.11. *Note that the result of the previous Lemma means that $g(x,T)$ is uniformly equivalent to $g(x,0)$.*

Corollary 4.12. *Let $(M, g(t))$ be a solution of the Ricci flow on a closed manifold. If*

$$|Rm(x,t)|_g \leq K$$

for all $x \in M$ and $t \in [0, T)$ (with $T < \infty$), then the metric $g(t)$ converges uniformly to a continuous metric $g(T)$ such that for all $x \in M$

$$e^{-C} g(x,0) \leq g(x,T) \leq e^{C} g(x,0).$$

Proof. A bound on $|\mathrm{Rm}(x,t)|_g$ implies one on $|\mathrm{Ric}(x,t)|_g$ and hence on $\left|\frac{\partial}{\partial t} g(t,x)\right|_{g(t)}$ by the Ricci flow equation. The integral of the previous lemma is then an integral of a bounded quantity over a finite interval and hence is bounded. Hence the lemma applies. □

The strategy for proving Theorem 4.9 will be the following: If we assume that $|\mathrm{Rm}(x,t)|_g \leq K$ and show that the metric converges uniformly to a smooth metric $g(T)$, then we can apply Theorem 2.11 with initial metric $g(T)$ to extend the solution past T. This would contradict the choice of T as the maximal time such that the Ricci flow exists on $[0, T)$. Let's see what we have so far. We have shown that there exists a limit metric $g(T)$ and it is continuous. We still need to prove that this metric is smooth. To do this, we need to make sure that the spatial derivatives of g near the limit time T are not blowing-up, i.e. that they are bounded.

The first thing we will do, is to bound the derivatives of curvature via the Bernstein-Bando-Shi estimates, which give bounds on the derivatives of the curvature under assumptions of bounded curvature. Recall that the Bernstein-Bando-Shi estimates give us no control for the derivatives of the curvature at $t = 0$. However this is no problem for us, because we are interested in derivative estimates near $t = T$.

Corollary 4.13 (of Proposition 4.3). *Let $(M^n, g(t))$ be a solution of the Ricci flow on a compact manifold. If there exist $\beta, K > 0$, such that*

$$|Rm(x,t)|_{g(t)} \leq K,$$

for all $t \in [0,T]$, where $T > \beta/K$, then for each $m \in \mathbb{N}$ there exists a constant B_m depending only on m, n and $min\{\beta, 1\}$ such that

$$|\nabla^m Rm(x,t)|_{g(t)} \leq B_m K^{1+\frac{m}{2}},$$

for all $t \in \left[\frac{min\{\beta,1\}}{K}, T\right]$.

Proof. The proof uses the Bernstein-Bando-Shi estimates and can be found in [1] pg. 202. □

We have now a bound for the derivatives of the curvature near $t = T$. We would like to use it in order to bound the derivatives of the metric near $t = T$ (we need this in order to show uniform convergence of $g(T)$ in any C^k norm).

Corollary 4.14. *Let $(M^n, g(t))$ be a solution of the Ricci flow on a closed manifold and let (x^i), $i = 1, ..., n$ be a local coordinate system defined on some coordinate chart $U \subset M$. If there exists $K > 0$ such that*

$$|Rm(x,t)|_{g(t)} \leq K,$$

for all $t \in [0,T)$, then for each $m \in \mathbb{N}$ there exist constants C_m, $C_m^{'}$ depending only on the chosen coordinate chart such that

$$|\partial^m g(x,t)| \leq C_m$$

and

$$|\partial^m Ric(x,t)| \leq C_m^{'},$$

for all $(x,t) \in U \times [0,T)$, where the norms are taken with respect to the Euclidean metric in the coordinate system (x^i).

Remark 4.15. *The previous Corollary is a reformulation of the argument in* [1] *pg. 203. This formulation is suggested in* [6] *pg. 54. Note that by $\partial^m g$ we mean the $(m+2,0)$-tensor field, defined only in the coordinate chart U, which has coordinates $\partial_{i_1} ... \partial_{i_m} g_{pq}$ with respect to the coordinate system (x^i). The Euclidean metric, which is also defined only in U, is the metric which has coordinates δ_{ij} with respect to the coordinate system (x^i).*

Proof. The proof is very technical and lengthy. It uses the Corollary 4.13 and can be found in [6] pg. 54 and [1] pg. 203. □

Corollary 4.16. *The metric $g(T)$ of Corollary 4.12 is smooth and the metrics $g(t)$ converge uniformly in every C^k norm to $g(T)$ as $t \to T$.*

Proof. This is again a quite technical proof. The convergence part of the Corollary is proved by applying Corollary 4.14. The proof can be found in [6] pg. 55. □

We are now in position to prove Theorem 4.9.

Proof. We assume for a contradiction that $|\mathrm{Rm}(x,t)|_g \leq K$. By the Corollaries above we know that the metrics $g(t)$ converge uniformly in any C^k norm to a smooth metric $g(T)$. Because $g(T)$ is smooth it is possible to find a solution to the Ricci flow with initial metric $g(T)$ by the result of Theorem 2.11. Thus our solution to the Ricci flow can be extended past $t = T$. This extension is smooth, because all spatial derivatives are continuous at $t = T$ (by the convergence of $g(t)$ in any C^k norm). It follows that all space-time derivatives are continuous at $t = T$ because the Ricci flow equation allows us to write time derivatives of quantities related to the metric in therms of space derivatives of those quantities and the space-derivatives have been shown to be continuous. Therefore, the solution can be extended past time $t = T$, so the time T could not have been maximal. This is a contadiction, □

5. Three-manifolds with positive Ricci curvature

The aim of this chapter is to demonstrate the following Theorem proved by Hamilton in [4].

Theorem 5.1. *Let M^3 be a closed manifold, which admits a smooth Riemannian metric with strictly positive Ricci curvature. Then M^3 also admits a smooth metric of constant positive curvature.*

Remark 5.2. *In particular if M^3 is simply connected, then M^3 is diffeomorphic to $\mathbb{S}^3$.*

5.1. Finite-time Blow-up.

The short time existence and uniqueness result of Theorem 2.11 guarantees that the Ricci flow has a unique solution on a maximal time interval $[0, T)$. We will show, that if the initial Ricci curvature is strictly positive, then $T < \infty$. The results of this Chapter can be found in [1] Chapter 6.8.

Theorem 5.3. *Let $(M, g(t))$ be a solution of the Ricci flow on a compact manifold, defined for $t \in [0, T)$. If the metric $g_0 = g(0)$ has strictly positive scalar curvature (in particular, if it has strictly positive Ricci curvature), then $g(t)$ becomes singular in finite time, i.e. $T < \infty$.*

Proof. M is compact and at $t = 0$ the scalar curvature is strictly positive, thus it is bounded from below by some $\rho > 0$. By using the evolution equation for the scalar curvature we obtain

$$\frac{\partial}{\partial t} R = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{n} R^2.$$

We now apply the maximum principle of Theorem 3.1. The solution of the ODE

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{2}{n}\phi^2 \\ \phi(0) &= p \end{aligned}$$

is $\phi(t) = \frac{1}{\frac{1}{\rho} - \frac{2t}{n}}$. Thus by Theorem 3.1 $R(x,t) \geq \phi(t)$. But $\phi(t)$ clearly diverges to $+\infty$ in finite time, hence $R(x,t)$ becomes singular and so the solution $g(t)$ becomes singular in finite time. □

Corollary 5.4. *The curvature blows-up as* $t \to T$*:*

$$\lim_{t\to T}\Big(\sup_{x\in M^3}|Rm(x,t)|\Big)=\infty$$

$$\lim_{t\to T}\Big(\sup_{x\in M^3}|Ric(x,t)|\Big)=\infty.$$

Proof. The previous Theorem states that the maximal time $T<\infty$. Thus by Theorem 4.9 the curvature blows-up as we approach the singular time T. Since $|\mathrm{Rm}|\le C|\mathrm{Ric}|$ in dimension $n=3$ the second statement follows as well. □

5.2. **The Uhlenbeck Trick.** We would like to apply the maximum principle for vector bundles (Theorem 3.2) to the Riemann curvature tensor. Recall that by Lemma 4.2, (3) the evolution of the Riemann curvature tensor under the Ricci flow is given by:

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl}+2(B_{ijkl}-B_{ijlk}-B_{iljk})\\ &\quad -(R_i^pR_{pjkl}+R_j^pR_{ipkl}+R_k^pR_{ijpl}+R_l^pR_{ijkp}),\end{aligned}$$

where $B_{ijkl}=-R^q_{pij}R^p_{qlk}$. The naive approach would be to apply Theorem 3.2 to R_{ijkl} by interpreting it as a section of the vector bundle of $(4,0)$-tensors. Unfortunately, the problems in that case would be two. The first one is that the reaction terms of the expression above are not useful when we try to solve the associated ODE in order to apply Theorem 3.2. The second problem is that the maximum principle for vector bundles cannot deal with bundle metrics that depend on time. In this chapter we examine a trick attributed to Karen Uhlenbeck ([5], pg. 155) that allows one to simplify the above equation by removing the last collection of terms with a "change of variables". This Chapter is based on [1] Chapters 6.2-6.4.

Let $(M,g(t))$, $t\in[0,T)$ be a solution to the Ricci flow with $g(0)=g_0$. Let V be vector bundle over M isomorphic to TM and let $\iota_0: V\to TM$ be a bundle isomorphism. In other words the restrictions $(\iota_0)_x: V_x\to T_xM$ are vector space isomorphisms depending smoothly on $x\in M$. Then we define a metric h_0 on V by

$$h_0:=\iota_0^*(g_0)$$

and we automatically obtain a bundle isometry

$$\iota_0:(V,h_0)\to(TM,g_0).$$

Corresponding to the evolution equation of the metric $g(t)$ by the Ricci flow, we evolve the isometry $\iota(t)$ by

$$\begin{aligned}\frac{\partial}{\partial t}\iota &= \mathrm{Ric}\circ\iota \qquad (5.1)\\ \iota(0) &= \iota_0.\end{aligned}$$

Here we regard the Ricci tensor $\mathrm{Ric}=\mathrm{Ric}(g(t))$ as a $(1,1)$-tensor. For each $x\in M$ we obtain a system of linear ODEs. Hence a unique solution exists for $t\in[0,T)$ (namely for as long as the solution $g(t)$ of the Ricci flow exists). Clearly $\iota(t)$ remains a smooth bundle isomorphism for all $t\in[0,T)$. But more is true.

Lemma 5.5. *Define* $h(t):=\iota(t)^*g(t)$. *Then the bundle maps*

$$\iota(t):(V,h(t))\to(TM,g(t))$$

remain isometries.

Proof. $\iota(t) : (V, h(t)) \to (TM, g(t))$ is an isometry as long as $h(t) = \iota(t)^* g(t)$. Because h is constant and ι_0 is an isometry by definition, it suffices to show that $\iota(t)^* g(t)$ does not change in time. Let $x \in M$ and $X, Y \in V_x$. Then

$$\begin{aligned}
\frac{\partial}{\partial t} h(X,Y) &= \frac{\partial}{\partial t}\Big((\iota^* g)(X,Y)\Big) \\
&= \frac{\partial}{\partial t}\Big(g(\iota(X), \iota(Y))\Big) \\
&= \Big(\frac{\partial}{\partial t} g\Big)(\iota(X), \iota(Y)) + g\Big(\frac{\partial}{\partial t}\iota(X), Y\Big) + g\Big(X, \frac{\partial}{\partial t}\iota(Y)\Big) \\
&= -2\mathrm{Ric}(\iota(X), \iota(Y)) + g(\mathrm{Ric}(\iota(X)), Y) + g(X, \mathrm{Ric}(\iota(Y))) \\
&= 0.
\end{aligned}$$

□

Therefore $\iota^* g(t)$ is independent of time t and so continues to equal the fixed metric h_0. Now we want to consider the pullback of Rm by ι. Let $(e_a)_{a=1}^n$ be a basis of sections of V restricted to an open set $U \subset M$ Then the components (R_{abcd}) of $\iota^*\mathrm{Rm}$ are

$$R_{abcd} = (\iota^*\mathrm{Rm})(e_a, e_b, e_c, e_d).$$

Lemma 5.6. *If $g(t)$ is a solution of the Ricci flow and $\iota(t)$ a solution of (5.1), then $\iota^* Rm$ evolves by*

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}).$$

Proof. The proof can be found in [1] pg. 182. □

Recall that we can regard Rm as a bilinear form

$$\mathcal{R} : \Lambda^2 T_x M \times \Lambda^2 T_x M \to \mathbb{R},$$

hence as a section of the bundle $S^2(\Lambda^2 T^* M)$, such that

$$\mathcal{R}(e_i \wedge e_j, e_k \wedge e_l) = R_{ijkl}.$$

If $\dim M = 3$, then $\dim(\Lambda^2 T_x M) = 3$ and if $\{E_1, E_2, E_3\}$ is an orthonormal basis for $T_x M$ then $\{E_1 \wedge E_2, E_1 \wedge E_3, E_2 \wedge E_3\}$ is an orthonormal basis for $\Lambda^2 T_x M$. Of course $\mathcal{R}$ can be represented by a symmetric 3×3 matrix, which can be diagonalized with respect to an orthonormal basis. We would like to apply the vector bundle maximum principle (Theorem 3.2) to $\mathcal{R} \in \Gamma(S^2(\Lambda^2 T^* M))$. We must consider the ODE corresponding to the PDE which describes the evolution of Rm of the previous Lemma. This is

$$\frac{d}{dt}\mathcal{R}_{abcd} = 2(B_{abcd}(\mathcal{R}) - B_{abdc}(\mathcal{R}) + B_{acbd}(\mathcal{R}) - B_{adbc}(\mathcal{R})).$$

In dimension three this ODE has a particularly convenient form. If we choose a basis so that $\mathcal{R}_0$ is diagonal with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$, then the equation is given by:

$$\frac{d}{dt}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 + \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_3\lambda_1 & 0 \\ 0 & 0 & \lambda_3^2 + \lambda_1\lambda_2 \end{bmatrix}.$$

In particular $\mathcal{R}(t)$ will remain diagonal, which is not in general true in higher dimensions. A more explicit description can be found in [1] Chapter 6.3, 6.4.

So in dimension three λ_1, λ_2 and λ_3 completely describe $\mathcal{R}$ and we may represent $\mathcal{R}$ as a point $(\lambda_1, \lambda_2, \lambda_3)$ moving in $\mathbb{R}^3$ according to the ODE

$$\frac{d}{dt}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = \begin{bmatrix}\lambda_1^2+\lambda_2\lambda_3\\ \lambda_2^2+\lambda_3\lambda_1\\ \lambda_3^2+\lambda_1\lambda_2\end{bmatrix}. \tag{5.2}$$

From the standard Riemannian geometry theory (you can look up [6] pg. 18) observe that the initial value for λ also tells us about the initial Ricci and scalar curvature, which means

$$\mathrm{Ric} = \frac{1}{2}\begin{bmatrix}\lambda_2+\lambda_3 & 0 & 0\\ 0 & \lambda_3+\lambda_1 & 0\\ 0 & 0 & \lambda_1+\lambda_2\end{bmatrix} \tag{5.3}$$

and

$$R = \lambda_1+\lambda_2+\lambda_3. \tag{5.4}$$

Remark 5.7. *Observe that* $\lambda_3 =$ *two times the sectional curvature of* $E_1 \wedge E_2 = Rm(E_1, E_2, E_1, E_2) = \mathcal{R}(E_1 \wedge E_2, E_1 \wedge E_2)$.

5.3. **The Local Pinching Estimates.** We present the pinching results which are true for 3-manifolds with positive Ricci curvature. As we will see, the first estimate proves that curvature pinching in preserved, whereas the second shows that it improves, hence that a solution to the Ricci flow on a 3-manifold with positive Ricci curvature in nearly Einstein at any point where its scalar curvature is large. These are pointwise estimates. Later we will demonstrate the techniques for comparing curvatures at different points of a solution. This chapter is based on [1] Chapter 6.5.

Lemma 5.8 (Ricci Pinching is Preserved)**.** *Let* $(M^3, g(t))$ *be a solution of the Ricci flow on a closed* 3*-manifold such that the initial metric* g_0 *has strictly positive Ricci curvature. If there exists constants* $C < \infty$ *and* $\varepsilon > 0$ *such that*

$$\frac{\lambda_1}{\lambda_3+\lambda_2} \le C$$

and

$$Ric \ge \varepsilon g$$

at $t = 0$*, then these inequalities persist as long as the solution exists.*

Proof. The proof can be found in [1] pg. 189 and 190. We present here an outline:

- C, ε exist at $t = 0$ by the compactness of M: $\frac{\lambda_1}{\lambda_3+\lambda_2}$ is a continuous, positive function at $t = 0$, thus there exists an upper bound for it. The eigenvalues of Ric at $t = 0$ are strictly positive, so they have an upper bound. It suffices to show that these conditions are preserved under the Ricci flow.
- We apply the maximum principle for vector bundles (Theorem 3.2) on $E = S^2(\Lambda^2 T^*M)$ with

  $K = \{Q \in E : \lambda_1(Q) - C(\lambda_3(Q)+\lambda_2(Q)) \le 0 \text{ and } \lambda_1+\lambda_3 \ge 2\varepsilon\}.$
- We show that K is invariant under parallel translation and convex in each fiber.
- Show that the solution of the associated ODE stays in K. That is if $\frac{\lambda_1}{\lambda_3+\lambda_2} \le C$ and $\lambda_2+\lambda_3 \ge 2\varepsilon$ initially, then this condition remains true under (5.2).

- Because the assumptions of Theorem 3.2 are fulfilled, the PDE stays in K, i.e. the conditions are preserved.

□

Corollary 5.9. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold such that the initial metric g_0 has strictly positive Ricci curvature. Then the scalar curvature blows-up as $t \to T$. That is, if we define $R_{\max}(t) := \sup_{x\in M} R(x,t)$, then*

$$\lim_{t\to T} R_{\max}(t) = \infty.$$

Proof. By the previous Lemma Ric remains strictly positive under the Ricci flow. This means that if we diagonalize Ric with respect to some orthonormal basis with eigenvalues $a, b, c > 0$, then

$$|\mathrm{Ric}|^2 = a^2 + b^2 + c^2 < (a+b+c)^2 = R^2.$$

The results follows from Theorem 5.4.. □

Corollary 5.10. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold such that the initial metric g_0 has strictly positive Ricci curvature. Then these exists a constant $\beta > 0$ depending only on g_0 such that at all points of M*

$$Ric \geq 2\beta^2 Rg.$$

Proof. We use the previous Lemma and the formulas (5.3) and (5.4).

$$\mathrm{Ric} \geq \frac{\lambda_2 + \lambda_3}{2} g \geq \frac{\lambda_1}{2C} g \geq \frac{\lambda_1 + \lambda_2 + \lambda_3}{6C} g \geq \frac{1}{6C} Rg.$$

□

Theorem 5.11 (Ricci Pinching is improved)**.** *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold such that the initial metric g_0 has strictly positive Ricci curvature. Then there exist positive constants $\delta < 1$ and C depending only on g_0 such that*

$$\frac{\lambda_1 - \lambda_3}{R} \leq \frac{C}{R^\delta}.$$

Proof. The proof can be found in [1] pg. 190-192. We present here an outline:

- It suffices to show that
$$\frac{\lambda_1 - \lambda_3}{\lambda_2 + \lambda_3} \leq \frac{\hat{C}}{(\lambda_2 + \lambda_3)^\delta}.$$
- By compactness given $\delta \in (0,1)$ we can choose a $\hat{C}$ at $t = 0$, because $\lambda_2 + \lambda_3 > 0$ by the Ric > 0 condition. We will show that this condition in preserved under the Ricci flow.
- Apply the maximum principle of Theorem 3.2 on $E = S^2(\Lambda^2 T^* M)$ with
$$K = \{Q \in E : (\lambda_1(Q) - \lambda_3(Q)) - \hat{C}(\lambda_2(Q) - \lambda_3(Q))^{1-\delta} \leq 0\}.$$
- We show that K is invariant under parallel translation and convex in each fiber.
- Show that the solution of the associated ODE stays in K by using equation (5.2). In this part we also use equation 5.2 and Lemma 5.8.
- Because the assumptions of Theorem 3.2 are fulfilled, the PDE stays in K, i.e. the conditions are preserved.

□

The previous Theorem states the following: As $t \to T$ and the curvature blows-up, the sectional curvatures get pinched together. Obviously $\lambda_1 - \lambda_3$ is the greatest difference between any two eigenvalues. Furthermore the left hand side is scale invariant, so even if when we rescale the metric by some factor, this bound tells us that the eigenvalues (i.e. the sectional curvatures) will be close together.

The previous Theorem is equivalent to the following Corollary, which was proved in Hamilton's original paper.

Corollary 5.12. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold such that the initial metric g_0 has strictly positive Ricci curvature. Then there exist positive constants B, $\hat{\delta}$ such that*

$$\frac{|E|^2}{R^2} \leq BR^{-\hat{\delta}},$$

where E is the traceless Ricci tensor $E_{ij} = R_{ij} - \frac{1}{3}Rg$,

Proof. We present an outline of the proof:

- We compute the matrix for E by (5.3) and (5.4).
- We prove the result by applying the previous Theorem for $\hat{\delta} = 2\delta$.

□

The previous Corollary reflects the discussion in the beginning of the chapter. $|E|^2$ measures how far away the metric is from being an Einstein metric. Recall that a metric is called Einstein if its traceless Ricci tensor is identically zero. When $|E|^2 = 0$ we have that $R_{ij} = Cg_{ij}$, where C is a constant over the whole manifold. In the 3-dimensional case if follows that the metric has constant sectional curvature.

5.4. **The Global Curvature Estimates.** In this chapter we obtain a gradient estimate for the scalar curvature. This estimate is important because it enables us to compare curvatures at different points, whereas the pinching estimate of the previous chapter is a pointwise estimate which compares sectional curvatures at the same point. We know that they pinch together if the scalar curvature blows-up at that point, but we only know that the curvature explodes somewhere on our manifold as we approach the singular time. This is not enough to conclude that the sectional curvatures pinch together everywhere.

Recall formula $\frac{|E|^2}{R^2} \leq B \cdot R^{-\hat{\delta}}$ from Corollary 5.12. We know from the classical Riemannian geometry theory, that if the metric is Einstein, i.e. $E = 0$, then the scalar curvature is constant. So if we have a bound on $|E|^2$ like the previous one everywhere on the manifold, then the scalar curvature might be close to being constant. So it is reasonable to expect that we will be able to obtain a bound on $|\nabla R|$ from our pinching result. A bound like this would allow us to compare values of R at different points of the manifold. But we already know that R is blowing-up somewhere on M, so we will be able to show that it is getting large everywhere and hence the sectional curvatures are getting close together everywhere.

Our references are [1] Chapter 6.6, Chapter 6.8 and [6] Chapter 7.5.

Theorem 5.13. *Let $(M, g(t))$ be a solution of the Ricci flow on a closed 3-manifold with $g(0) = g_0$. If $Ric(g_0) > 0$, then there exist $\bar{\beta}, \gamma > 0$ depending only on g_0 such that for any $\beta \in [0, \bar{\beta}]$ there exists C such that*

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\gamma} + CR^{-2}.$$

Here, the left-hand side is a scale invariant quantity, while the right-hand side is small when the scalar curvature is large.

Proof. The proof can be found in [1] pg. 194. The proof uses the maximum principle of Theorem 3.1 for the evolution equation of $\frac{|\nabla R|^2}{R}$. □

We apply now the gradient estimate and the Bonnet-Myers Theorem to show that the global pinching of the curvature tends to 1.

Theorem 5.14. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold with $g(0) = g_0$. If $Ric(g_0) > 0$, then there exist constants $C, \gamma > 0$ depending only on g_0 such that*

$$\frac{R_{\min}}{R_{\max}} \geq 1 - CR_{\max}^{-\gamma}.$$

Note that this means $R_{\min}/R_{\max} \to 1$ as $t \to T$, because $R_{\max} \to \infty$ as $t \to T$ by Corollary 5.9. It follows that $R \to \infty$ uniformly as $t \to T$.

Proof. The proof can be found in [1] pg. 210.We present an outline:

- The main tool is the previous Theorem, which allows us to compare the curvature at different points of M.
- By Corollary 4.9 $R_{\max} \xrightarrow{t \to T} \infty$. Thus for t sufficiently close to T the previous theorem tells us, that there exist positive constants A and α, such that $|\nabla R| \leq AR_{\max}^{3/2-\alpha}$. Thus for t sufficiently close to T we have
$$|R(x) - R(y)| \leq \int_\gamma |\nabla R| ds \leq AR_{\max}^{3/2-\alpha} d(x, y),$$
where γ is the minimizing geodesic connecting x and y.
- Let us choose $x(t)$ such that $R_{\max}(t) = R(x, t)$ (we can do this as M is compact) and define $L(t) := \frac{1}{\varepsilon\sqrt{R_{\max}(t)}}$, where $\epsilon > 0$. Then for all $y \in B(x(t), L(t))$ we have
$$R(y) \geq R(x) - AR_{\max}^{3/2-\alpha} L \geq R_{\max}(1 - \frac{A}{\varepsilon} R_{\max}^{-\alpha}).$$
Recall that $R_{\max} \to \infty$ as $t \to T$. Therefore given $\delta > 0$ for t sufficiently close to T we have
$$R(y) \geq (1 - \delta)R_{\max},$$
for all $y \in B(x(t), L(t))$.
- By using the Bonnet-Myers Theorem we show that $B\Big(x(t), \frac{1}{\varepsilon\sqrt{R_{\max}(t)}}\Big)$ is all of M.

□

Now because $R \to \infty$ uniformly as $t \to T$ the curvature should be getting uniformly pinched.

Corollary 5.15. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold with $g(0) = g_0$ with $Ric(g_0) > 0$. Furthermore let $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ denote the eigenvalues of $\mathcal{R}$ at (x, t). Then for any $\varepsilon \in (0, 1)$ there exists $T_\varepsilon \in [0, T)$ such that*

$$\min_{x \in M} \lambda_3(x, t) \geq (1 - \varepsilon) \max_{y \in M} \lambda_2(y, t) > 0$$

for all $t \in [T_\varepsilon, T)$. This means that the metric will eventually attain positive sectional curvature everywhere.

Proof. The proof can be found in [1] pg. 210. We present an outline:

- We apply Theorem 5.11:
$$\lambda_3 \geq \lambda_1 - \hat{C}(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta} \geq \lambda_1(1 - 3\hat{C}R^{-\delta})$$
to each point $x \in M$.
- By the previous Theorem $R \to \infty$ uniformly as $t \to T$. Thus if we are given $\eta > 0$ for some time t close enough to T, we can show that
$$\lambda_3(x,t) \geq (1-\eta)^3 \lambda_1(y,t).$$
- The result follows by taking the supremum over all $x, y \in M$.

□

5.5. **The Normalized Ricci Flow.** Let's sum up what we have so far. The Ricci flow becomes singular is some finite time T. As we approach the singular time T the curvature blows-up and the sectional curvatures get globally pinched together as the curvature blows up. We want a metric of constant sectional curvature on M, so we should take the limit of the flow as $t \to T$. But the manifold is shrinking to a point at time T. We should consider of rescaling so that the volume of M remains constant (i.e. the volume of M with respect to $\tilde{g}$ is constant).

In order to define our new, normalized flow we define dilating factors $\psi(t) > 0$ so that the metrics $\tilde{g}(t) = \psi(t) \cdot g(t)$with $\psi(0) = 1$ have constant volume. The resulting evolution equation is
$$\frac{\partial}{\partial t}\tilde{g} = -2\tilde{\mathrm{Ric}} + \frac{2\tilde{r}}{n} g,$$
where $\tilde{\mathrm{Ric}}$ denotes the Ricci curvature of $\tilde{g}$ and $\tilde{r} := \frac{\int_M \tilde{R} d\mu}{\int_M d\mu}$ the average scalar curvature of $\tilde{g}$. Furthermore $\tau = \int_0^t \psi(u) du$ corresponds to a rescaling of time. Note that n is the dimension of the manifold. This is the equation of the normalized Ricci flow and differs from the unnormalized Ricci flow by a rescaling of space and time. Now the metric has constant volume and the problem of the manifold shrinking to a point as $t \to T$ is eliminated. Because we have only rescaled the Ricci flow solution, the results that we have proven so far for the unnormalized Ricci flow can be translated to the normalized Ricci flow. Further details on the construction of the normalized Ricci flow equation can be found in [1] Chapter 6.9. Now the issue of the curvature exploding is fixed by the following Lemma:

Lemma 5.16. *For the normalized Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature there exists some positive constant C, so that $\tilde{R}_{\max} < C$.*

Proof. The proof can be found in [1] Chapter 6.9. □

We will now mention a very important result, which states that the normalized flow exists for all time. In other words our rescaling of space and time has in fact taken us to an infinite time interval,

Theorem 5.17. *Let $(M, g(t))$ be a solution of the unnormalized Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature. Then the corresponding normalized solution exists for all time, i.e. $T = \infty$.*

Proof. The proof can be found in [1] Chapter 6.9. We present an outline:

- We show that
$$\int_0^T R_{\max}(t)dt = \infty,$$
where $R_{\max}(t)$ is the maximal scalar curvature of the metric $g(t)$ and $[0, T)$ is the maximal time interval on which the unnormalized flow exists.
- The corresponding integral for the normalized flow will be the same:
$$\int_0^{\tilde{T}} \tilde{R}_{\max}(\tau)d\tau = \int_0^T r(t)dt = \infty.$$
- But the integrand is bounded by the previous Lemma. So $\tilde{T} = \infty$.

□

5.6. **Exponential Convergence of the Normalized Flow.** In this chapter we follow [6] Chapter 7.7.

We will prove that the normalized Ricci flow converges as $\tau \to \infty$ to a smooth metric $\tilde{g}_\infty$ of constant positive sectional curvature. We will use the following notation:
$$\tilde{g}_\infty := \lim_{\tau\to\infty} \tilde{g}(\tau).$$
In what follows Lemma 4.10 will play a crucial role. In order to show that $\tilde{g}_\infty$ exists and is continuous, we have to show that there exists some $C < \infty$ such that
$$\int_0^\infty \left|\frac{\partial}{\partial\tau}\tilde{g}\right|_{\tilde{g}} d\tau < C.$$
If we use the normalized Ricci flow equation, then it is equivalent to showing that the integral
$$\int_0^\infty \left|\tilde{\text{Ric}} - \frac{\tilde{r}}{3}\tilde{g}\right|_{\tilde{g}} d\tau \tag{5.5}$$
is bounded. The easiest way to do it, is to show that the integrand is bounded by a decaying exponential.

The next one is the key theorem in order to prove that the integral (5.5) is bounded.

Theorem 5.18. *If $(M, \tilde{g}(\tau))$ is a solution of the normalized Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature, then there exist constants $C, \delta > 0$ such that*
$$|\tilde{E}| \le Ce^{-\delta\tau}$$

Proof. The proof can be found in [6] Chapter 7.8. □

To apply Lemma 4.10 we must prove an exponential bound on $\left|\tilde{\text{Ric}} - \frac{\tilde{r}}{3}\tilde{g}\right|$. The previous Theorem gives us a bound on $|\tilde{E}| = \left|\tilde{\text{Ric}} - \frac{\tilde{R}}{3}\tilde{g}\right|$, which is almost what we want. We just need to go from $\tilde{R}$ to $\tilde{r}$. So it suffices to show that the difference $|\tilde{R} - \tilde{r}|$ is exponentially bounded. We will actually prove something a bit stronger:

Lemma 5.19. *There exist constants $C, \delta > 0$ such that*
$$\tilde{R}_{\max} - \tilde{R}_{\min} < Ce^{-\delta\tau}.$$

Proof. The proof can be found in [6] Chapter 7.8. □

This allows us to prove the following:

Theorem 5.20. *Let $(M, \tilde{g}(\tau))$ be a solution of the normalized Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature. Then $\tilde{g}(\tau)$ exists for all $\tau \in [0, \infty)$ and converges uniformly as $\tau \to \infty$ to a continuous metric $\tilde{g}_\infty$.*

Proof. By the theorem and lemma above we have

$$\begin{aligned} \int_0^\infty \left|\frac{\partial \tilde{g}}{\partial \tau}\right| d\tau &= \int_0^\infty \left|\tilde{\mathrm{Ric}} - \frac{\tilde{r}}{3}\tilde{g}\right| d\tau \\ &\leq \left|\tilde{\mathrm{Ric}} - \frac{\tilde{R}}{3}\tilde{g}\right| + \left|\frac{\tilde{R} - \tilde{r}}{3}\tilde{g}\right| d\tau \\ &< \int_0^\infty Ce^{-\delta\tau} d\tau < \infty, \end{aligned}$$

where we have amalgamated the two exponential bounds into one. It follows by Lemma 4.10 that $\tilde{g}(\tau)$ converges uniformly to a continuous metric $\tilde{g}_\infty$ as $\tau \to \infty$. □

But we want our limit metric $\tilde{g}_\infty$ to be smooth. So the next thing to prove is that convergence is smooth. Another important reason that we require smoothness is that it will allow us to conclude that the curvature pinching results we have proven for the flow lead to similar results for the limit metric and hence that the limit metric has constant curvature.

Theorem 5.21. *The limit metric $\tilde{g}_\infty$ of the previous Theorem is smooth and the convergence $\tilde{g}(\tau) \xrightarrow{\tau\to\infty} \tilde{g}_\infty$ in uniform in every C^k norm.*

Proof. The proof can be found in [6] Chapter 7.8. □

Finally we are in position to prove Theorem 5.1.

Theorem 5.22. *The limit metric $\tilde{g}_\infty$ is a smooth metric with constant positive sectional curvature.*

Proof. By the previous Theorem $\tilde{g}(\tau)$ converges to $\tilde{g}_\infty$ in the C^0, C^1 and C^2 norms. Because all the curvature quantities are combinations of 0-th order, 1-st order and 2-nd order derivatives of the metric, this means that we can take the limit to show that the traceless Ricci tensor of $\tilde{g}_\infty$ vanishes:

$$|\tilde{E}_\infty| = \lim_{\tau\to\infty} |\tilde{E}(\tau)| \leq \lim_{\tau\to\infty} Ce^{-\delta\tau} = 0.$$

Therefore $\tilde{g}_\infty$ is Einstein and thus has constant (positive) sectional curvature. □

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CHAOTIC LINEAR OPERATORS

FOTIS THOMOS

ABSTRACT. n this master thesis we study linear maps, which are chaotic, according to the definition that was suggested by Devaney in 1986: an operator is chaotic if it has an element with dense orbit and if it has a dense set of periodic points. We mention three classical examples of linear operators - Birkhoff 's translation operator, MacLane's differentiation operator and Rolewicz's backward shift operator - and we present the proof that these operators are chaotic.

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HIERARCHY AND EXPANSIVENESS IN 2-DIMENSIONAL SUBSHIFTS OF FINITE TYPE

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Abstract. Using a deterministic version of the self-similar method for constructing 2-dimensional subshifts of finite type (SFTs), we construct aperiodic 2D SFTs with a unique direction of non-expansiveness and prove that the emptiness problem of SFTs is undecidable even in this restricted case. As an additional application of our method, we characterize the sets of directions that can be the set of non-expansive directions of 2D SFTs.

1. Alphabets and Configurations

- **Alphabet**: A finite set of letters $\mathcal{A}$.
- (dD) **Configuration**: A mapping $c\colon \mathbb{Z}^d \to \mathcal{A}$.
- (dD) **Full shift**: The set of all configurations $\mathcal{A}^{\mathbb{Z}^d}$.
- **Compact topology** for the full shift.
- $(c_i)_{i\in\mathbb{N}}$ converges iff $(c_i(\vec{x}))_{i\in\mathbb{N}}$ is eventually constant for all $\vec{x} \in \mathbb{Z}^d$.
- **Shift action**: $\sigma^{\vec{n}}(c)(\vec{x}) = c(\vec{x}+\vec{n})$, $\forall c \in \mathcal{A}^{\mathbb{Z}^d}, \forall \vec{x} \in \mathbb{Z}^d$.

2. Subshifts of Finite Type

- **Pattern**: A partial assignment $p\colon D \to \mathcal{A}$, where $D \subseteq \mathbb{Z}^d$ is *finite*.
- **Subshift**: A subset $X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ defined by the set of forbidden patterns $\mathcal{F}$.
- $X_{\mathcal{F}} = \{c \in \mathcal{A}^{\mathbb{Z}^d} : \sigma^{\vec{n}}(c)_{|D} \neq p\}$, for all $\vec{n} \in \mathbb{Z}^d$ and all patterns $p\colon D \to \mathcal{A}$ in $\mathcal{F}$.
- Subshifts are the closed and σ-invariant subsets of $\mathcal{A}^{\mathbb{Z}^d}$.
- **Subshift of Finite Type (SFT)**: A subshift defined by a *finite* set of forbidden patterns.

3. Aperiodic SFTs

- **Periodic configuration** c: $\exists \vec{n} \in \mathbb{Z}^d$ such that $c(\vec{x}+\vec{n}) = c(\vec{x})$, $\forall \vec{x} \in \mathbb{Z}^d \iff \sigma^{\vec{n}}(c) = c$.
- **Aperiodic SFT**: Non-empty, but does not contain a periodic configuration.
- A non-empty 1D SFT always contains a periodic configuration. No aperiodic 1D SFT.

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- What happens in higher dimensions?

Theorem 3.1 (Berger 1966, Robinson 1971, Kari-Culik 1995, Jeandel-Rao 2015)**.** *There exists an aperiodic 2D SFT.*

4. Undecidability of the Emptiness Problem

The Emptiness Problem Given a dD finite set of forbidden patterns $\mathcal{F}$, is $X_{\mathcal{F}} \neq \emptyset$?

- The Emptiness Problem is decidable for $d = 1$.
- **Graph representation** of 1D SFTs.
- What happens in higher dimensions?

Theorem 4.1 (Berger 1966, Robinson 1971, Kari 2008)**.** *The Emptiness Problem is undecidable for $d = 2$.*

5. Expansive and Non-Expansive Directions

- X is a $2D$ subshift.
- $l \in \mathbb{R} \sqcup \{\infty\}$ is a **slope**.
- $\ell \subset \mathbb{R}^2$ is the corresponding **line** through the origin.
- $\ell_r \subset \mathbb{R}^2$ is the corresponding **stripe** of width $2r$.

Definition l is **expansive** for X if there exists $r > 0$ such that every $x \in X$ is determined by x_{ℓ_r}.

- A 2D configuration encoded in a 1D strip.
- A subshift can have many expansive directions.
- $\mathcal{N}(X)$ denotes the set of *non*-expansive directions of X.

6. Extremely Expansive Subshifts

Theorem 6.1 (Boyle-Lind 1997)**.** *$\mathcal{N}(X) \neq \emptyset$ if and only if X is infinite.*

- X finite $\Rightarrow \mathcal{N}(X) = \emptyset$.
- X infinite $\Rightarrow \mathcal{N}(X) \neq \emptyset$.
- **Extremely expansive**: $|\mathcal{N}(X)| = 1$.
- Most restricted non-trivial case.
- Being extremely expansive is a strong geometric restriction.
- Reducing a 2D object to 1D as much as possible.
- Are extremely expansive 2D SFTs closer to the 1D or to the 2D case?

7. Aperiodicity and Undecidabilty for Extremely Expansive SFTs

Theorem 7.1 (Guillon-Z. 2016)**.** *There exists an aperiodic* extremely expansive *2D SFT.*

Theorem 7.2 (Guillon-Z. 2016)**.** *The Emptiness Problem is undecidable for* extremely expansive *2D SFTs.*

- Previously, known for $|\mathcal{N}(X)| = 2$ (Kari-Papasoglou 1999, Lukkarila 2008). However, a new method is needed to give extremely expansive SFTs.

Structure of $\mathcal{N}(X)$ for general subshifts Question What can $\mathcal{N}(X) \subseteq \mathbb{R} \sqcup \{\infty\}$ look like?

Theorem 7.3 (Boyle-Lind 1997)**.** $\mathcal{N}(X)$ *is closed under the one-point compactification of* $\mathbb{R} \sqcup \{\infty\}$.

Theorem 7.4 (Hochman 2011)**.** *For every closed set of directions* $\mathcal{N}_0$*, there exists a subshift* X *such that* $\mathcal{N}(X) = \mathcal{N}_0$.

Structure of $\mathcal{N}(X)$ for 2D SFTs

Theorem 7.5. $\mathcal{N}(X)$ *is* effectively *closed under the one-point compactification of* $\mathbb{R} \sqcup \{\infty\}$.

- There exists an algorithm that discards directions not in $\mathcal{N}(X)$.
- Additional computational theoretic restriction.
- As happens usually in 2D SFTs, necessary computational restriction turns out to be also sufficient.

Theorem 7.6 (Guillon-Z. 2016)**.** *For every* effectively *closed set of directions* $\mathcal{N}_0$*, there exists a* 2*D SFT* X *such that* $\mathcal{N}(X) = \mathcal{N}_0$.

A few words about the construction technique

- Hierarchical (or fixed-point, or self-similar) construction.
- Introduced by Gacs to solve the Positive Rates Conjecture.
- Used in an SFT context by Durand-Romashchenko-Shen.
- We use a vertically expansive version of their method.
- For the last result, we modify the construction of Hochman.

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